

# Optimal Controller for a Time-Dependent System

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Presented by Academician A.B. Kurzhanskii July 11, 2001

Received July 18, 2001

1. Let  $T = [t_*, t^*]$ ,  $0 \leq t_* < t^* < +\infty$ ,  $N$  be a positive integer,  $h = \frac{t^* - t_*}{N}$ ,  $A(t)$  and  $b(t)$  ( $t \in T$ ) be a piecewise continuous  $n \times n$  matrix function and a piecewise continuous  $n$ -vector function, and  $T_u = \{t_*, t_* + h, \dots, t^* - h\}$ . A function  $u(t)$  ( $t \in T$ ) is called a discrete control (with quantization step  $h$ ) if  $u(t) = u(t_* + kh)$ ,  $t \in [t_* + kh, t_* + (k + 1)h[$ ,  $k = 0, 1, \dots, N - 1$ .

In the class of discrete controls, we consider a family of linear problems depending on  $\sup(\tau, z)$  [1]:

$$\begin{aligned} c'x(t^*) \rightarrow \max, \quad \dot{x} &= A(t)x + b(t)u, \\ x(\tau) &= z, \quad Hx(t^*) = g, \\ |u(t)| &\leq 1, \quad t \in T(\tau) = [\tau, t^*], \quad x \in R^n, \\ u \in R, \quad g \in R^m, \quad \text{rank} H &= m < n. \end{aligned} \quad (1)$$

At  $\tau = t_*$  and  $z = x_0$ , this family includes the original problem of optimizing a time-varying system.

Let  $u^0(t|\tau, z)$ ,  $t \in T(\tau)$ , be the optimal open-loop control of problem (1) in the position  $(\tau, z)$ , and let  $X(\tau)$  be the set of states  $z$  at which problem (1) has a solution for a fixed  $\tau$ .

The function

$$u^0(\tau, z) = u^0(\tau|\tau, z), \quad z \in X(\tau), \quad \tau \in T_u \quad (2)$$

is called an optimal feedback control.

The control system of problem (1) supplemented with feedback (2) becomes a closed-loop system. Suppose that the behavior of the closed-loop system under persistent piecewise continuous disturbances  $w(t)$  ( $t \in T$ ) is described by the equation

$$\dot{x} = A(t)x + b(t)u^0(t, x) + w(t), \quad x(t_*) = x_0. \quad (3)$$

The trajectory of system (3) is a solution to the equation

$$\begin{aligned} \dot{x} &= A(t)x + b(t)u(t) + w(t), \\ x(t_*) &= x_0, \quad u(t) = u^0(t_*, kh, x(t_* + kh)), \\ t \in [t_* + kh, t_* + (k + 1)h[ \quad &k = 0, 1, \dots, N - 1. \end{aligned}$$

Consider a particular control process with an initial state  $x(t_*) = x_0^*$  and a disturbance  $w^*(t)$ ,  $t \in T$ . They correspond to a trajectory  $x^*(t)$  ( $t \in T$ ) of system (3) that satisfies the identity

$$\begin{aligned} \dot{x}^*(t) &\equiv A(t)x^*(t) + b(t)u^0(t, x^*(t)) + w^*(t), \\ t \in T, \quad x(t_*) &= x_0^*. \end{aligned}$$

We say that (i)  $u^*(t) = u^0(t, x^*(t))$  ( $t \in T_u$ ) is a realization of the optimal feedback (2); (ii)  $u^*(t)$  ( $t \in T_u$ ) is constructed in real time if the time required for calculating  $u^*(\tau) = u^0(\tau, x^*(\tau))$  in every current position  $(\tau, x^*(\tau))$  does not exceed  $h$ ; (iii) this is done by a device called an optimal controller.

2. Assume that an operation algorithm for an optimal controller has been designed, and the optimal controller produced signals  $u^*(t_*)$ ,  $u^*(t_* + h)$ ,  $\dots$ ,  $u^*(\tau)$  at the times  $t_*$ ,  $t_* + h$ ,  $\dots$ ,  $\tau$ , respectively. Let  $x^*(\tau + h)$  be the system state at  $\tau + h$  resulting from the action of these signals and the current disturbance  $w^*(t)$ ,  $t \in [t_*, \tau + h[$ . According to (2), the open-loop solution  $u^0(t|\tau + h, x^*(\tau + h))$ , to problem (1) at  $(\tau + h, x^*(\tau + h))$  is required for the controller to calculate  $u^*(\tau + h)$ . By assumption, the optimal controller has constructed  $u^0(t|\tau, x^*(\tau))$  [ $t \in T(\tau)$ ] at the point  $\tau$ . A part of this solution  $u^0(t|\tau, x^*(\tau))$ ,  $t \in T(\tau + h)$ , is the optimal open-loop control at  $(\tau + h, x^0(\tau + h))$  resulting from the action of  $u^0(\tau|\tau, x^*(\tau))$  at  $(\tau, x^*(\tau))$  in the absence of any disturbance ( $w(t) = 0$ ,  $t \in [\tau, \tau + h[$ ). The actual state  $x^*(\tau + h)$  is related to the ideal state  $x^0(\tau + h)$  by

$$x^*(\tau + h) = x^0(\tau + h) + \int_{\tau}^{\tau + h} F(\tau + h)F^{-1}(s)w^*(s)ds.$$

For small  $h > 0$  and bounded  $w^*(t)$ ,  $t \in [\tau, \tau + h[$ , the vectors  $x^*(\tau + h)$  and  $x^0(\tau + h)$  are nearly identical; therefore, the controller can generate  $u^0(t|\tau + h, x^*(\tau + h))$ ,

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$t \in T(\tau + h)$  by correcting the already known control  $u^0(t|\tau, x^*(\tau))$ ,  $t \in T(\tau + h)$ . Below, we describe the basic steps of the correction procedure proposed.

In functional form, problem (1) for the position  $(\tau, x^*(\tau))$  takes the form

$$\sum_{t \in T_u(\tau)} c_h(t)u(t) \rightarrow \max, \quad \sum_{t \in T_u(\tau)} d_h(t)u(t) = g(\tau),$$

$$|u(t)| \leq 1, \quad t \in T_u(\tau);$$

$$c_h(t) = \int_t^{t+h} \psi'_c(\vartheta)b(\vartheta)d\vartheta, \tag{4}$$

$$d_h(t) = \int_t^{t+h} G(\vartheta)b(\vartheta)d\vartheta, \quad g(\tau) = g - G(\tau)x^*(\tau);$$

$T_u(\tau) = \{\tau, \tau + h, \dots, t^* - h\}$ , and  $\psi_c(t)$  and  $G(t)$  [ $t \in T(\tau)$ ] are the solutions to the equations

$$\dot{\psi} = -A'(t)\psi, \quad \psi(t^*) = c;$$

$$\dot{G} = -GA(t), \quad G(t^*) = H. \tag{5}$$

The set  $T_{\text{sup}}(\tau) = \{t, l = 1, 2, \dots, m\} \subset T_u(\tau)$  is called a support if the matrix  $D_{\text{sup}}(\tau) = (d_h(t), t \in T_{\text{sup}}(\tau))$  is nonsingular [2–4]. The pair  $\{u_\tau(\cdot), T_{\text{sup}}(\tau)\}$  consisting of the admissible control  $u_\tau(\cdot) = (u(t), t \in T(\tau))$  and a support is called a support control.

The support  $T_{\text{sup}}(\tau)$  corresponds to the  $m$ -vector of potentials  $v(\tau)$ :  $v'(\tau)D_{\text{sup}}(\tau) = c'_{\text{sup}}(\tau)$ ,  $c_{\text{sup}}(\tau) = (c_h(t), t \in T_{\text{sup}}(\tau))$ , the cocontrol

$$\Delta_h(t) = c_h(t) - v'(\tau)d_h(t)$$

$$= \int_t^{t+h} (\psi'_c(\vartheta) - v'(\tau)G(\vartheta))b(\vartheta)d\vartheta, \quad t \in T_u(\tau),$$

and the pseudocontrol  $\omega(t)$ ,  $t \in T_u(\tau)$ :

$$\omega(t) = \begin{cases} -1 & \text{if } \Delta_h(t) < 0 \\ 1 & \text{if } \Delta_h(t) > 0; \end{cases}$$

$$\omega(t) \in [-1, 1] \quad \text{if } \Delta_h(t) = 0,$$

$$t \in T_n(\tau) = T_u(\tau) \setminus T_{\text{sup}}(\tau),$$

The support values of  $\omega(t)$  [ $t \in T_{\text{sup}}(\tau)$ ] satisfy the equation

$$\sum_{t \in T_{\text{sup}}(\tau)} d_h(t)\omega(t) + \sum_{t \in T_n(\tau)} d_h(t)\omega(t) = g(\tau).$$

The pseudocontrol  $\omega^0(t)$  [ $t \in T_u(\tau)$ ] corresponding to the optimal support  $T_{\text{sup}}^0(\tau)$  is the optimal open-loop control  $u^0(t|\tau, x^*(\tau))$ ,  $t \in T(\tau)$ , in problem (4) [3]. Thus,

to generate the signal  $u^*(\tau + h)$  at  $\tau + h$ , it is sufficient to know the optimal support  $T_{\text{sup}}^0(\tau + h)$ . Let us describe the operations transforming the optimal support  $T_{\text{sup}}^0(\tau)$  into  $T_{\text{sup}}^0(\tau + h)$ .

3. To simplify the presentation, we assume that the condition of dual nondegeneracy holds for any support  $T_{\text{sup}}(\tau)$ ,  $\tau \in T_u$ :

$$\Delta_h(t) \neq 0, \quad t \in T_n(\tau); \quad \Delta_h(t-h)\Delta_h(t+h) < 0$$

$$\text{if } \tau < t < t^* - h, \quad t \in T_{\text{sup}}(\tau);$$

$$\Delta_h(\tau + h) \neq 0 \quad \text{if } \tau \in T_{\text{sup}}(\tau);$$

$$\Delta_h(t^* - 2h) \neq 0 \quad \text{if } t^* - h \in T_{\text{sup}}(\tau)$$

(the general case was analyzed in [3].) A point  $t \in T_n(\tau)$  is called a nonsupport zero if  $\Delta_h(t-h)\Delta_h(t) < 0$ . Let  $T_{n0}(\tau)$  be the set of nonsupport zeros and  $T_{\text{sn}}(\tau) = T_{\text{sup}}(\tau) \cup T_{n0}(\tau) \cup \{\tau, t^*\} = \{t_k, k \in K(\tau) = k^*(\tau) + 1\}$ ,  $K(\tau) = \{0, 1, \dots, k^*(\tau)\}$ . Denote by  $T_k(\tau)$ ,  $k \in K(\tau)$ , the intervals where the cocontrol has a definite sign:

$$T_k(\tau) = \{t_{*k} = t_k, t_k + h, \dots, t_k^* = t_{k+1} - h\}$$

$$\text{if } t_k \notin T_{\text{sup}}(\tau);$$

$$T_k(\tau) = \{t_{*k} = t_k + h, t_k + 2h, \dots, t_k^* = t_{k+1} - h\}$$

$$\text{if } t_k \in T_{\text{sup}}(\tau).$$

If  $t^* - h \in T_{\text{sup}}(\tau)$ , we set  $T_{k^*}(\tau) = \emptyset$ .

Assume that the following data have been obtained by solving problem (1) at position  $(\tau, x^*(\tau))$  and have been stored:

- (i) an optimal support  $T_{\text{sup}}^0(\tau)$ ;
- (ii) the set  $T_{n0}(\tau)$ ;
- (iii) the support matrix  $D_{\text{sup}}(\tau)$  and the vector  $d_h(\tau)$ ;
- (iv)  $G(t)$  and  $\psi_c(t)$  at  $t \in T_{\text{on}}(\tau) \setminus t^*$ ;
- (v)  $u^*(\tau)$ ,  $\gamma(\tau) = u^*(\tau)$  if  $\tau \notin T_{\text{sup}}^0(\tau)$ , and  $\gamma(\tau) = \text{sgn} \Delta_h^\tau(\tau + h)$  if  $\tau \in T_{\text{sup}}^0(\tau)$ ;
- (vi) the vector

$$p(\tau) = \gamma(\tau) \sum_{k=0}^{k^*(\tau)} (-1)^k \sum_{t \in T_k(\tau)} d_h(t)$$

$$= \gamma(\tau) \sum_{k=0}^{k^*(\tau)} (-1)^k \int_{t_{*k}}^{t_k^*+h} G(\vartheta)b(\vartheta)d\vartheta;$$

- (vii) the vector of potentials  $v(\tau)$ .

The problem to be solved by the controller at  $\tau + h$  has the form

$$\begin{aligned} \sum_{t \in T_u(\tau+h)} c_h(t)u(t) &\rightarrow \max, \\ \sum_{t \in T_u(\tau+h)} d_h(t)u(t) &= g(\tau+h), \quad (6) \\ |u(t)| &\leq 1, \quad t \in T_u(\tau+h). \end{aligned}$$

We calculate  $u^*(\tau+h) = u^0(\tau+h|\tau+h, x^*(\tau+h))$  by using the data obtained at  $\tau$ . We distinguish the following two cases: (i)  $\tau \notin T_{\text{sup}}^0(\tau)$ , and (ii)  $\tau \in T_{\text{sup}}^0(\tau)$ .

In case (i), the initial values at  $\tau+h$  are set to be  $T_{\text{sup}}(\tau+h) := T_{\text{sup}}^0(\tau)$ ,  $T_{n0}(\tau+h) := T_{n0}(\tau)$ ,  $D_{\text{sup}}(\tau+h) := D_{\text{sup}}(\tau)$ , and  $v(\tau+h) := v(\tau)$ . Integrating Eqs. (5) over  $[\tau, \tau+h]$  with the initial conditions  $\psi_c(\tau)$  and  $G(\tau)$ , we obtain and store  $\psi_c(\tau+h)$  and  $G(\tau+h)$ ; set  $p(\tau+h) = p(\tau) - \gamma d_h(\tau)$ ; compute and store the vector  $d_h(\tau+h)$ ; and set  $\gamma(\tau+h) := \gamma(\tau)$  if  $\tau+h \notin T_{\text{sup}}(\tau+h)$ ;  $\gamma(\tau+h) := -\gamma(\tau)$  if  $\tau+h \in T_{\text{sup}}(\tau+h)$ ;  $T_{n0}(\tau+h) = T_{n0}(\tau)$  if  $\tau+h \notin T_{n0}(\tau)$ ;  $T_{n0}(\tau+h) = T_{n0}(\tau) \setminus \{\tau+h\}$  if  $\tau+h \in T_{n0}(\tau)$ ;  $D_{\text{sup}}(\tau+h) = D_{\text{sup}}(\tau)$ , and  $v(\tau+h) = v(\tau)$ . If  $\tau+h \notin T_{\text{sup}}(\tau)$ , then  $T_{\text{sup}}(\tau+h) = T_{\text{sup}}(\tau) \setminus \tau \cup \tau+h$  and  $k^*(\tau+h) = k^*(\tau)$ . If  $\tau+h \in T_{\text{sup}}(\tau)$ , then  $T_{\text{sup}}(\tau+h) = T_{\text{sup}}(\tau) \setminus \tau$  and  $k^*(\tau+h) = k^*(\tau) - 1$ .

The iteration process transforming the current support  $T_{\text{sup}}(\tau+h)$  is performed by steps. For simplicity, we set  $T_{\text{sup}} = T_{\text{sup}}(\tau+h)$ ,  $T_n = T_n(\tau+h)$ ,  $T_{\text{sup}} = T_{\text{sup}}(\tau+h)$ ,  $T_{n0} = T_{n0}(\tau+h)$ ,  $D_{\text{sup}} = D_{\text{sup}}(\tau+h)$ ,  $v = v(\tau+h)$ ,  $p = p(\tau+h)$ ,  $T_k = T_k(\tau+h)$ ,  $k^* = k^*(\tau+h)$ , and  $\gamma = \gamma(\tau+h)$ .

**Preliminary step.** From the equation  $D_{\text{sup}}\omega_{\text{sup}} = g(\tau+h) - p$ , we find  $\omega(t)$ ,  $t \in T_{\text{sup}}$ . If

$$|\omega(t)| \leq 1, \quad t \in T_{\text{sup}}, \quad (7)$$

then we set  $T_{\text{sup}}^0(\tau+h) = T_{\text{sup}}$ , feed the signal  $u^*(\tau+h) = \gamma(\tau+h)$  if  $\tau+h \notin T_{\text{sup}}^0$ ; or  $u^*(\tau+h) = \omega(\tau+h)$  if  $\tau+h \in T_{\text{sup}}^0$  as input to system (3), and go to the position  $(\tau+2h, x^*(\tau+2h))$ .

If (7) does not hold, then we choose  $t^0$  from  $T_{\text{sup}}$  such that  $|\omega(t^0)| = \max|\omega(t)|$  at  $t \in T_{\text{sup}}$  and find the vector  $\Delta v$ :  $-D'_{\text{sup}}\Delta v = (\Delta\delta_h(t), t \in T_{\text{sup}})$ , where  $\Delta\delta_h(t^0) = \text{sgn}\omega(t^0)$  and  $\Delta\delta_h(t) = 0, t \in T_{\text{sup}} \setminus t^0$ .

Define the function

$$\begin{aligned} \delta_h(t, \sigma) &= \Delta_h(t) + \sigma\Delta\delta_h(t), \quad t \in T_u(\tau+h), \\ \sigma &\geq 0 \quad (\text{varied cocontrol}), \end{aligned} \quad (8)$$

where

$$\Delta\delta_h(t) = -\Delta v' d_h(t) = - \int_t^{\tau+h} \Delta v' G(\vartheta) b(\vartheta) d\vartheta, \quad t \in T_n$$





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