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Optimization of the Multidimensional Control Systems with Parallelepiped Constraints[1]

R. Gabasov*, N. M. Dmitruk, and F. M. Kirillova****

*Belarussian State University, Minsk, Belarus

**Institute of Mathematics, Belarussian National Academy of Sciences, Minsk, Belarus

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Abstract—The general problem of optimization of the linear nonstationary multidimensional systems with parallelepiped control constraints was considered. Two problems of designing the optimal program controls and optimal feedback controls were solved. The proposed methods rely on a new realization of the adaptive method of linear programming. The results were illustrated by examples.

1. INTRODUCTION

The theory of optimal control passed a long way over the last fifty years. Its main advances are concerned with the profound results of the qualitative theory [1]. The issues of the constructive theory are less studied [2], which is confirmed by the fact that, despite the abundance of the existing algorithms, it is difficult to select an adequate method of designing the optimal program controls even for the linear problems, to say nothing about the design of optimal feedback controls.

The present paper suggests to use linear programming as the basis of the proposed methods of program and positional optimization of the linear nonstationary systems. Linear programming [3] was the first part of the modern theory of extremal problems which differs fundamentally from the classical theory of extremal problems of the 1950's in that it methodically takes into consideration the constraints in the form of inequalities. Historically, however, the advent and development of the theory of optimal control was not related with linear programming. This theory was treated as a nonclassical stage of the variational calculus. The problems, theory, and methods of linear programming were regarded as extremely bounded and unable to exceed the narrow confines of the simple economic problems. This accounts for the increased interest of the mathematical theory of optimal processes in the traditional qualitative issues of the variational calculus to the detriment of the constructive issues which in the 1940–1950's were not regarded by the mathematicians as topical and had no powerful computers to support them. In linear programming, on the contrary, the issues of actual (numerical) solution were at the foreground.

After discovering the main facts of the theory of optimal control, attempts were made to make use of the simplex method of linear programming to solve numerically the problems of optimal control [4]. Unfortunately, these and other efforts were not carried to their conclusion, although it was known how efficient the simplex method can be if one makes the best use of its opportunities for solving special linear problems [3].

The present paper aims at developing realizations of the adaptive method of linear programming, which was proposed in Minsk at the early 1980's [5], that would take into consideration at the most the dynamic structure of the problem of optimal control. Being aware of the fact that without discrete computers numerical solution of the problems of optimal control is impossible, the present

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authors decided to optimize the continuous systems using the discrete controls with finite time-slotting period, that is, the problem of optimal control is solved in the class of discrete controls, which allows one to bypass some analytic problems and make direct use of the apparatus of linear programming. As can be seen from the results obtained, adequate consideration of the dynamic nature of the problems at hand enables one to construct very fast algorithms to tackle them. This fact is especially important for designing the feedback optimal controls. This problem, which is pivotal to the theory of optimal control, was formulated in the pioneer works on the optimal control as early as in the 1940–1950's [6, 7]. However, it still remains unsolved in the classical (analytic) form even for the linear problems. The constructive approach to the problem of optimal design [8] relies on the fast algorithms. Therefore, the present paper can be regarded as a proof of the possibility of using them to design optimal systems of sufficiently high orders.

Section 2 formulates the general linear problem of optimization of the multidimensional nonstationary control systems with regard for the parallelepiped constraints. Section 3 considers the static method of solution. Section 4 introduces the support, a basic element of the method, which is used to formulate the principles of maximum and ε-maximum (Section 5). Section 6 describes the direct algorithm to construct the suboptimal open-loop controls. Section 7 presents the dual method which plays an important part in the design of the optimal systems (Section 8). Section 9 describing a numerical experiment concludes the paper. One can see from it the extent of efficiency of the proposed methods. Methods of constructing the optimal open-loop controls (in the example below) in two integrations of the dynamic system, as well as methods of designing the optimal feedback controls are lacking in the existing literature. The findings of the paper are new for the stationary systems as well.

2. FORMULATION OF THE PROBLEM

Let $T = [t_*, t^*]$; $h = (t^* - t_*)/N$, $t_* < t^* < +\infty$, N be a natural number, and $T_u = \{t_*, t_* + h, \dots, t^* - h\}$. The function $u(t) = (u_j(t),\ j \in J)$, $t \in T$, $J = \{1, 2, \dots, r\}$, will be called the discrete r-dimensional control with the time-slotting period h, provided that $u_j(t) \equiv u_j(t_* + kh)$, $j \in J$, $t \in [t_* + kh, t_* + (k+1)h[\,$, $k = \overline{0, N-1}$.

Let us consider the linear problem of terminal control in the class of discrete controls:

$$
\begin{gathered}
c'x(t^*) \to \max, \\
\dot{x} = A(t)x + B(t)u, x(t_*) = x_0, \\
g_* \le Hx(t^*) \le g^*, \quad u(t) \in U, \quad t \in T,
\end{gathered}
\tag{1}
$$

where $x = x(t) \in R^n$ is the state of the control system at the instant t; $u = u(t) \in R^r$ is the value of control at time t; $A(t) \in R^{n \times n}$ and $B(t) = (b_j(t),\ j \in J) \in R^{n \times r}$, $t \in T$, are the sectionally continuous matrix functions; $b_j(t)$ is the jth column of the matrix $B(t)$; g_*, $g^* \in R^m$, $H \in R^{m \times n}$; $H' = (h_{(i)},\ i \in I)$, $h_{(i)}$ is the n-vector, the ith row of the matrix H, $I = \{1, 2, \dots, m\}$; and U is the set of accessible values of control. It is assumed below that the set U is a parallelepiped: $U = \{u \in R^r : u_* \le u \le u^*\}$. The constructions below are not appreciably affected in the case of the nonstationary set $U(t) = \{u \in R^r : u_*(t) \le u \le u^*(t)\}$, $t \in T$.

The discrete control $u(t)$, $t \in T$, of system (1) and its corresponding trajectory $x(t)$, $t \in T$, will be referred to as admissible if they satisfy the constraints on problem (1).

The admissible control $u^0(t)$, $t \in T$, and the trajectory $x^0(t)$, $t \in T$, will be referred to as optimal (program solution of problem (1)) if along them the performance index attains its maximum:

$$
c'x^0(t^*) = \max_u c'x(t^*).
$$

For the given $\varepsilon \geq 0$, the suboptimal (ε-optimal) control $u^\varepsilon(t)$, $t \in T$, and trajectory $x^\varepsilon(t)$, $t \in T$, are defined by the inequality

$$c'x^0(t^*) - c'x^\varepsilon(t^*) \leq \varepsilon.$$

To introduce the notion of positional solution (optimal feedback control), we embed problem (1) into the family of problems

$$\begin{aligned} &c'x(t^*) \to \max, \quad \dot{x} = A(t)x + B(t)u, \quad x(\tau) = z, \\ &g_* \leq Hx(t^*) \leq g^*, \quad u(t) \in U, \quad t \in T(\tau) = [\tau, t^*], \end{aligned} \tag{2}$$

which depends on the scalar $\tau \in T_u$ and n-vector z.

Let $u^0(t \,|\, \tau, z)$, $t \in T(\tau)$, be the optimal open-loop control of problem (2) for the position (τ, z) and X_τ be the set of states z for which (2) has the optimal program solutions. According to the theory of optimal processes [1], the function

$$u^0(\tau, z) = u^0(\tau \,|\, \tau, z), \quad z \in X_\tau, \quad \tau \in T_u, \tag{3}$$

will be called the optimal (discrete) feedback control (positional solution of problem (1)), and construction of function (3) will be called the design of the optimal feedback (design of the optimal system). Replacement of the control in (1) by function (3) is called the closing of the control system. Under constant perturbation $w(t)$, $t \in T$, the trajectory of the closed system

$$\dot{x} = A(t)x + B(t)u^0(t, x) + w(t), \quad x(t_*) = x_0,$$

is the solution of equation $\dot{x} = A(t)x + B(t)u^0(t) + w(t)$, $x(t_*) = x_0$, $u^0(t) \equiv u^0(t_* + kh, x(t_* + kh))$, $t \in [t_* + kh, t_* + (k+1)h[$, $k = \overline{0, N-1}$. The present paper aims at describing the algorithms to construct the program and positional solutions of (1).

3. "STATIC" METHOD OF CONSTRUCTING THE PROGRAM SOLUTIONS

The simplest (static) method of solving the "dynamic" extremal problem (1) lies in reducing it to that of linear programming. It has been known [9] that the state $x(t)$ of the control system (1) can be calculated using the Cauchy formula

$$x(t) = F(t, t_*)x_0 + \int_{t_*}^{t} F(t, \tau)B(\tau)u(\tau)d\tau, \tag{4}$$

where $F(t, \tau) = F(t)F^{-1}(\tau)$, $F(t) \in R^{n \times n}$, $\dot{F} = A(t)F$, $F(0) = E$.

By substituting (4) in (1), we obtain an equivalent functional form [5] of (1) in the class of discrete controls:

$$\begin{aligned} &\sum_{t \in T_u} c'(t)u(t) \to \max, \\ &\widetilde{g}_* \leq \sum_{t \in T_u} D(t)u(t) \leq \widetilde{g}^*, \\ &u_* \leq u(t) \leq u^*, \quad t \in T_u, \end{aligned} \tag{5}$$

where

$$c'(t) = (c_j(t), j \in J)' = \int_t^{t+h} c'F(t^*, \tau)B(\tau)d\tau,$$

$$D(t) = (d_j(t), j \in J) = (d_{ij}(t), i \in I, j \in J) = \int_t^{t+h} HF(t^*, \tau)B(\tau)d\tau, \quad t \in T_u; \tag{6}$$

$$\widetilde{g}_* = g_* - HF(t^*, t_*)x_0, \quad \widetilde{g}^* = g^* - HF(t^*, t_*)x_0.$$

The problem of linear programming (5) has m basic constraints, the interval inequalities, rN variables, and a dense matrix $(D(t),\ t \in T_u)$ of conditions. If the time-slotting period h is long enough, then problem (5) can be effectively handled by the standard methods of linear programming. For smaller time-slotting periods h, problem (5) becomes "semilarge" (with highly increased number of variables), and its solution requires a large area of the main memory. Another feature of (5) lies in that for close values of t, $\tau \in T_u$, the $(m+1)$-vectors $(c_j(t),\ d_j(t))$ and $(c_j(\tau), d_j(\tau))$ become almost collinear for each $j \in J$, which, for example, adversely affects the simplex method.

Now, the aim stated in Section 2 can be formulated as follows: *it is required to develop methods for solving problem* (5) *with small time-slotting periods h which take into account at the most its dynamic nature and are little sensitive to the value of h.* The methods are based on the special method of linear programming [5] adapted to the dynamic nature of problem (5).

We first present the dynamic methods of constructing elements (6) of problem (5). Let $\psi_c(t)$, $t \in T$, be a solution of the adjoint equation

$$\dot{\psi} = -A'(t)\psi \tag{7}$$

with the initial condition $\psi(t^*) = c$ and the $m \times n$-matrix function $G(t)$, $t \in T$, be the solution of the equation

$$\dot{G} = -GA(t) \tag{8}$$

with the initial condition $G(t^*) = H$. Then, elements of (6) can be constructed according to the formulas

$$c'(t) = \int_t^{t+h} \psi_c'(\tau)B(\tau)d\tau,\ D(t) = \int_t^{t+h} G(\tau)B(\tau)d\tau, \quad t \in T_u;$$

$$\widetilde{g}_* = g_* - G(t_*)x_0, \quad \widetilde{g}^* = g^* - G(t_*)x_0,$$

that is, $m+1$ processors construct the elements of (6) in one integration of the adjoint system over the interval T.

4. SUPPORT AND ITS ACCOMPANYING ELEMENTS

Let $I_s \subset I$ be an arbitrary nonempty subset with $s^* = |I_s|$ elements; $S = J \times T_u$ be the set of all possible pairs $\{j, t\}$ of the elements $j \in J$ and $t \in T_u$. We extract from S an arbitrary subset S_s with s^* elements and denote by $T_s(j) = \{t \in T_u : \{j, t\} \in S_s\}$, $j \in J$, the horizontal sections of the set S_s. Let us compile the matrix

$$D_s = \begin{pmatrix} d_{ij}(t), & \{j, t\} \in S_s \\ i \in I_s & \end{pmatrix}. \tag{9}$$

Definition 1 ([5]). The totality $K_{\rm s} = \{I_{\rm s},\ S_{\rm s}\}$ with $I_{\rm s} \neq \varnothing$, $S_{\rm s} \neq \varnothing$ is called the support of problem (1) if matrix (9) is nonsingular. By definition, the totality $K_{\rm s} = \{I_{\rm s} = \varnothing,\ S_{\rm s} = \varnothing\}$ is an (empty) support.

There are two ways to identify the nonempty support $K_{\rm s}$ by the elements of the original problem (1).

Direct method. Let us construct the terminal states $\chi_{k\tau}(t^*)$, $\{k,\tau\} \in S_{\rm s}$, of the first system (1) corresponding to the initial state $x(t_*) = 0$ and the controls

$$u_j(t) \equiv 0, \quad j \in J \setminus k, \quad t \in T;$$
$$u_k(t) = \begin{cases} 1, & t \in [\tau, \tau + h[\\ 0, & t \in T \setminus [\tau, \tau + h[\,. \end{cases}$$

Multiplication of these states by the matrix $H_{\rm s}$, $H'_{\rm s} = (h_{(i)},\ i \in I_{\rm s})$, provides the matrix $D_{\rm s} = (H_{\rm s}\chi_{k\tau}(t^*),\ \{k,\tau\} \in S_{\rm s})$. Therefore, the nonempty totality $K_{\rm s}$ is the support if and only if $\det D_{\rm s} \neq 0$.

Dual method. By integrating the adjoint system (7) with the initial conditions $\psi(t^*) = h_{(i)}$, we establish that $\xi_i(t)$, $t \in T$, $i \in I_{\rm s}$, and compile the matrix

$$D_{\rm s} = \begin{pmatrix} \int_t^{t+h} \xi'_i(\tau) b_j(\tau) d\tau, & \{j,t\} \in S_{\rm s} \\ i \in I_{\rm s} & \end{pmatrix}.$$

The fact that $K_{\rm s}$ is a support amounts to nondegeneracy of the matrix $D_{\rm s}$.

As one can see from the above methods of constructing the support matrix $D_{\rm s}$, the support can be identified by s^* parallel processors in one integration of the direct or adjoint system over the interval T.

In what follows, we use along with the support $K_{\rm s}$ its accompanying elements: (1) the vector of potentials $\nu = \nu(I) = (\nu_i,\ i \in I)$ (of the Lagrange multipliers accompanying the support); (2) cotrajectory $\psi(t) = (\psi_i(t),\ i = \overline{1,n})$, $t \in T$; (3) co-control $\Delta(t) = \Delta(t\,|\,J) = (\Delta_j(t),\ j \in J)$, $t \in T$; (4) pseudocontrol $\omega(t) = \omega(t\,|\,J) = (\omega_j(t),\ j \in J)$, $t \in T$, and pseudo-output $\zeta = \zeta(I) = (\zeta_i,\ i \in I)$; and (5) qiasicontrol $\widetilde{\omega}(t) = \widetilde{\omega}(t\,|\,J) = (\widetilde{\omega}_j(t),\ j \in J)$, $t \in T$, and the mismatch of the terminal constraints $g = g(I)$.

The potential vector ν is constructed using the following rules: $\nu_{\rm ns} = \nu(I_{\rm ns}) = (\nu_i,\ i \in I_{\rm ns}) = 0$, $I_{\rm ns} = I \setminus I_{\rm s}$; and $\nu_{\rm s} = \nu(I_{\rm s}) = (\nu_i,\ i \in I_{\rm s})$ is the solution of the vector equation

$$D'_{\rm s}\nu_{\rm s} = c_{\rm s},$$

where $c_{\rm s} = c(S_{\rm s}) = (c_j(t),\ \{j,t\} \in S_{\rm s}) = \left(\int_t^{t+h} \psi'_c(\tau) b_j(\tau) d\tau,\ \{j,t\} \in S_{\rm s}\right)$. In the case of nonempty support, we assume that $\nu = 0$.

We define the cotrajectory $\psi(t)$, $t \in T$, as the solution of the adjoint Eq. (7) with the initial condition $\psi(t^*) = c - H'\nu$. If $K_{\rm s} = \varnothing$, then $\psi(t^*) = c$.

We make use of the cotrajectory to introduce the co-control

$$\Delta'(t) = \int_t^{t+h} \psi'(\tau) B(\tau) d\tau, \quad t \in T_u,$$

featuring $\Delta_j(t) = 0$, $\{j,t\} \in S_{\rm s}$ [5].

To construct the pseudocontrol $\omega(t)$, $t \in T$, and the pseudo-output ζ, we first define the values of the nonsupport components $\omega_{\rm ns} = \omega(S_{\rm ns}) = (\omega_j(t),\ \{j,t\} \in S_{\rm ns})$, $S_{\rm ns} = S \setminus S_{\rm s}$, and the support components $\zeta_{\rm s} = \zeta(I_{\rm s})$:

$$\omega_j(t) = u_{*j}, \quad \text{if} \quad \Delta_j(t) < 0; \quad \omega_j(t) = u_j^*, \quad \text{if} \quad \Delta_j(t) > 0;$$
$$\omega_j(t) \in [u_{*j}, u_j^*], \quad \text{if} \quad \Delta_j(t) = 0; \quad \{j,t\} \in S_{\rm ns};$$
$$\zeta_i = g_{*i}, \quad \text{if} \quad \nu_i < 0; \quad \zeta_i = g_i^*, \quad \text{if} \quad \nu_i > 0;$$
$$\zeta_i \in [g_{*i}, g_i^*], \quad \text{if} \quad \nu_i = 0; \quad i \in I_{\rm s}.$$

In the case of empty support, we assume that $\zeta_{\rm s} = 0$.

The support component of the pseudocontrol $\omega_{\rm s} = \omega(S_{\rm s}) = (\omega_j(t),\ \{j,t\} \in S_{\rm s})$ is established from the equation system

$$\sum_{\{j,t\} \in S_{\rm ns}} d_{ij}(t)\omega_j(t) + \sum_{\{j,t\} \in S_{\rm s}} d_{ij}(t)\omega_j(t) = \zeta_i - h'_{(i)} F(t^*, t_*) x_0, \quad i \in I_{\rm s}. \tag{10}$$

If $K_{\rm s} = \varnothing$, then we assume that $\omega_{\rm s} = 0$.

In the case of a nonempty support, the dynamic method of constructing the vector $\omega_{\rm s}$ is as follows. Let $\varkappa_0(t^*)$ be at the instant t^* the value of the solution of the direct Eq. (1) with the initial condition $x(t_*) = x_0$ and control $u_j(t) = \omega_j(t)$, $\{j,t\} \in S_{\rm ns}$; $u_j(t) = 0$, $\{j,t\} \in S_{\rm s}$. Then with regard for notation (6), system (10) becomes as follows:

$$D_{\rm s}\omega_{\rm s} = \zeta_{\rm s} - H_{\rm s}\varkappa_0(t^*). \tag{11}$$

The solution $\varkappa(t)$, $t \in T$, of the direct Eq. (1) with the initial condition $x(t_*) = x_0$ and control $u(t) = \omega(t)$, $t \in T_u$, is called the pseudotrajectory (accompanying the support $K_{\rm s}$). By multiplying the pseudostate $\varkappa(t^*)$ by the matrix $H_{\rm ns}$, $H'_{\rm ns} = (h_{(i)},\ i \in I_{\rm ns})$, we get the nonsupport component $\zeta_{\rm ns} = \zeta(I_{\rm ns}) = H_{\rm ns}\varkappa(t^*)$ of the vector ζ.

We describe a special method of calculating the right side of Eq. (11) that will be used in the positional solution of problem (1). To simplify calculations, we assume that the co-control satisfies the conditions $\Delta_j(t-h)\Delta_j(t+h) < 0$ if $\{j,t\} \in S_{\rm s}$, $t_* < t < t^* - h$, $\Delta_j(t_* + h) \neq 0$ if $\{j, t_*\} \in S_{\rm s}$, and $\Delta_j(t^* - 2h) \neq 0$ if $\{j, t^* - h\} \in S_{\rm s}$.

The pair $\{j,t\} \in S_{\rm ns}$ will be called the nonsupport zero of the co-control if $\Delta_j(t-h)\Delta_j(t) < 0$. The set of all nonsupport zeros is denoted by $S_{\rm ns0}$, and its horizontal sections, by $T_{\rm ns0}(j) = \{t \in T_u : \{j,t\} \in S_{\rm ns0}\}$, $j \in J$. Let $T_0(j) = T_{\rm s}(j) \bigcup T_{\rm ns0}(j) \bigcup \{t_*, t^*\} = \{t_k(j), k \in K(j) \bigcup k(j)+1\}$, $K(j) = \{0, 1, \ldots, k(j)\}$; $T_k(j)$, $k \in K(j)$, be the intervals of fixed signs of the jth component of the co-control:

$$T_k(j) = \{t_{*k}(j) = t_k(j), t_k(j) + h, \ldots, t_k^*(j) = t_{k+1}(j) - h\}, \quad \text{if} \quad t_k(j) \notin T_{\rm s}(j);$$
$$T_k(j) = \{t_{*k}(j) = t_k(j) + h, t_k(j) + 2h, \ldots, t_k^*(j) = t_{k+1}(j) - h\}, \quad \text{if} \quad t_k(j) \in T_{\rm s}(j).$$

We introduce the numbers

$$\gamma_j = \begin{cases} \operatorname{sgn} \Delta_j(t_*), & \text{if} \quad t_* \notin T_{\rm s}(j) \\ \operatorname{sgn} \Delta_j(t_* + h), & \text{if} \quad t_* \in T_{\rm s}(j), j \in J; \end{cases}$$

$$\omega_j^k = \begin{cases} u_j^*, & \text{if} \quad (-1)^k \gamma_j > 0 \\ u_{*j}, & \text{if} \quad (-1)^k \gamma_j < 0, \quad k \in K(j), \quad j \in J \end{cases} \tag{12}$$

and vectors[2]

$$p_k(j) = \sum_{t \in T_k(j)} d_j(t) = \int\limits_{t_{*k}(j)}^{t_k^*(j)+h} G(\tau) b_j(\tau) d\tau, \quad k \in K(j), \quad j \in J;$$

$$p = H\varkappa_0(t^*) = G(t_*)x_0 + \sum_{j \in J} \sum_{k=0}^{k(j)} p_k(j)\omega_j^k.$$

For calculation of $\omega_{\rm s}$, Eq. (11) then takes the form

$$D_{\rm s}\omega_{\rm s} = \zeta_{\rm s} - p_{\rm s},$$

where $p_{\rm s} = (p_i,\ i \in I_{\rm s})$. By means of the vector p, one can also easily calculate the pseudo-output

$$\zeta = D_{|{\rm s}|}\omega_{\rm s} + p,$$

where $D_{|{\rm s}|} = (d_j(t),\ \{j,t\} \in S_{\rm s})$. The matrix $D_{|{\rm s}|}$ is constructed similar to $D_{\rm s}$,—for example, by the direct ($D_{|{\rm s}|} = (H\chi_{k\tau}(t^*),\ \{k,\tau\} \in S_{\rm s})$) or dual ($D_{|{\rm s}|} = \left(\int_t^{t+h} G(\tau)b_j(\tau)d\tau, \{j,t\} \in S_{\rm s}\right)$) method.

If the inequalities

$$u_{*j} \le \omega_j(t) \le u_j^*,\ \{j,t\} \in S_{\rm s}; \quad g_{*i} \le \zeta_i \le g_i^*, \quad i \in I_{\rm ns}, \tag{13}$$

are satisfied, then $u^0(t) = \omega(t)$, $t \in T_u$, is the optimal control.

By the quasicontrol accompanying the support $K_{\rm s}$ is meant the function

$$\widetilde{\omega}_j(t) = \begin{cases} \omega_j(t), & \text{if} \quad u_{*j} \le \omega_j(t) \le u_j^* \\ u_{*j}, & \text{if} \quad \omega_j(t) < u_{*j} \\ u_j^*, & \text{if} \quad \omega_j(t) > u_j^*; \quad j \in J, \quad t \in T_u. \end{cases}$$

In contrast to the pseudocontrol $\omega(t)$, $t \in T_u$, the direct constraints $u_* \le \widetilde{\omega}(t) \le u^*$, $t \in T_u$, are satisfied on the qiasicontrol $\widetilde{\omega}(t)$, $t \in T_u$, but the corresponding trajectory $\widetilde{x}(t)$, $t \in T$, of system (1) can violate the terminal constraints $g_i = g_{*i} - h'_{(i)}\widetilde{x}(t^*)$, $i \in I_- = \{i \in I : h'_{(i)}\widetilde{x}(t^*) < g_{*i}\}$; $g_i = h'_{(i)}\widetilde{x}(t^*) - g_i^*$, $i \in I_+ = \{i \in I : h'_{(i)}\widetilde{x}(t^*) > g_i^*\}$; $g_i = 0$, $i \in I_0 = \{i \in I : h'_{(i)}\widetilde{x}(t^*) \in [g_{*i}, g_i^*]\}$. The vector $g = (g_i,\ i \in I)$ is called the mismatch vector. The qiasicontrol is the optimal control for problem (1) with the terminal constraints $\underline{g}_* \le Hx(t^*) \le \overline{g}^*$, where $\underline{g}_* = (\underline{g}_{*i} = g_{*i},\ i \in I_0 \bigcup I_+;\ \underline{g}_{*i} = g_{*i} - g_i,\ i \in I_-)$, $\overline{g}^* = (\overline{g}_i^* = g_i^*,\ i \in I_0 \bigcup I_-;\ \overline{g}_i^* = g_i^* + g_i,\ i \in I_+)$.

With a knowledge of $\omega_{\rm s}$, we can readily calculate $\widetilde{\omega}_{\rm s} - \omega_{\rm s}$ and determine the quasi-output $\widetilde{\zeta} = H\widetilde{x}(t^*) = \zeta + D_{|{\rm s}|}(\widetilde{\omega}_{\rm s} - \omega_{\rm s})$ from which the mismatch vector g is calculated.

5. PRINCIPLES OF MAXIMUM AND ε-MAXIMUM

We formulate the criteria for optimality and suboptimality using the potential vector ν and the cotrajectory $\psi(t)$, $t \in T_u$, that accompany the support $K_{\rm s}$ [5].

Principle of maximum. For the admissible control $u(t)$, $t \in T$, and trajectory $x(t)$, $t \in T$, to be optimal, it is necessary and sufficient that there exists a support $K_{\rm s}$ such that the following conditions are satisfied on its accompanying potential vector ν and the cotrajectory $\psi(t)$, $t \in T$:

[2] If $t^* - h \in T_{\rm s}(j)$ for some $j \in J$, then we assume that $T_{k(j)}(j) = \varnothing$, $p_{k(j)}(j) = 0$.

(1) maximum condition for control:

$$\int_t^{t+h} \psi'(\tau)B(\tau)d\tau\ u(t) = \max_{u_* \le u \le u^*} \int_t^{t+h} \psi'(\tau)B(\tau)d\tau\ u, \quad t \in T_u;$$

(2) transversality condition for the trajectory

$$\nu' Hx(t^*) = \max_{g_* \le Hx \le g^*} \nu' Hx.$$

The support $K_{\rm s}$ that is used to identify the optimal open-loop control will be called the optimal support; it is accompanied by the optimal elements.

As can be seen from the construction of the pseudocontrol $\omega(t)$, $t \in T$, and qiasicontrol $\widetilde{\omega}(t)$, $t \in T$, each of these functions satisfies the maximum principle, but the direct constraints $u_* \le u(t) \le u^*$, $t \in T_u$, on the support components $\{j, t\} \in S_{\rm s}$ can be violated on the first function and the terminal constraints, on the second function.

Principle of $\boldsymbol{\varepsilon}$*-maximum.* For any $\varepsilon \ge 0$, for the admissible control $u(t)$, $t \in T_u$, and trajectory $x(t)$, $t \in T$, to be ε-optimal, it is necessary and sufficient that there exists a support $K_{\rm s}$ such that the following conditions are satisfied on its accompanying elements:

(1) condition for ε-maximum for control:

$$\int_t^{t+h} \psi'(\tau)B(\tau)d\tau\ u(t) = \max_{u_* \le u \le u^*} \int_t^{t+h} \psi'(\tau)B(\tau)d\tau\ u - \varepsilon_u(t), \quad t \in T_u;$$

(2) condition for ε-transversality for the trajectory:

$$\nu' Hx(t^*) = \max_{g_* \le Hx \le g^*} \nu' Hx - \varepsilon_x;$$

(3) condition for ε-accuracy:

$$\sum_{t \in T_u} \varepsilon_u(t) + \varepsilon_x \le \varepsilon.$$

As one can see from the above criteria, one integration over the interval T of the adjoint system with the corresponding potential vector suffices to identify the optimal and ε-optimal open-loop controls.

For the given $\varepsilon \ge 0$, $\delta \ge 0$, the admissible control $u(t) \in U$, $t \in T$, and the corresponding trajectory $x(t)$, $t \in T$, of system (1) are called the $\varepsilon\delta$-solution of problem (1), provided that the following inequalities are satisfied:

$$c'x^0(t^*) - c'x(t^*) \le \varepsilon, \|g\| \le \delta.$$

It follows from the principle of ε-maximum that for the admissible control $u(t)$, $t \in T$, and the trajectory $x(t)$, $t \in T$, to be the $\varepsilon\delta$-solution of problem (1), it is necessary and sufficient that there exists a support $K_{\rm s}$ such that the conditions for ε-maximum, ε-transversality, and ε-accuracy, as well as the inequality $\|g\| \le \delta$ are satisfied on its accompanying elements.

6. DIRECT METHOD OF CONSTRUCTING THE OPTIMAL OPEN-LOOP CONTROLS

In this paper, support is used not only to identify the optimal and suboptimal controls, but also as a main tool for constructing program and positional solutions. The methods proposed are iterative and aimed at constructing the $\varepsilon\delta$-solution of problem (1) for given numbers $\varepsilon \geq 0$ and $\delta \geq 0$. Since in the course of solving problem (1) the support is transformed along with the admissible control, it is only natural to consider them jointly.

We introduce the following definitions [10].

(1) The pair $\{u(\cdot), K_{\rm s}\}$ of the admissible control $u(\cdot) = (u(t), t \in T_u)$ and the support $K_{\rm s}$ is called the support control.

(2) The support control $\{u(\cdot), K_{\rm s}\}$ is directly nondegenerate if $u_{*j} < u_j(t) < u_j^*$, $\{j, t\} \in S_{\rm s}$.

(3) The support control $\{u(\cdot), K_{\rm s}\}$ is twice nondegenerate if the accompanying potential vector ν and the co-control $\Delta(t)$, $t \in T_u$, satisfy the relations $\nu_i \neq 0$, $i \in I_{\rm s}$; $\Delta_j(t) \neq 0$, $\{j, t\} \in S_{\rm ns}$.

(4) The number $\beta(u(\cdot), K_{\rm s}) = c'\varkappa(t^*) - c'x(t^*) = \sum\limits_{t \in T_u} \Delta'(t)(\omega(t) - u(t)) + \nu'(\zeta - Hx(t^*)) = \sum\limits_{t \in T_u} \varepsilon_u(t) + \varepsilon_x$ is called the suboptimality estimate of the support control $\{u(\cdot),\ K_{\rm s}\}$.

To simplify further calculations, we assume that only direct and twice nondegenerate support controls are used in the iterations of the method. The general case was considered in [5]. The iteration of the method is a replacement of the "old" support control $\{u(\cdot), K_{\rm s}\}$ by a "new" one $\{\overline{u}(\cdot), \overline{K}_{\rm s}\}$ for which the inequality $\beta(\overline{u}(\cdot), \overline{K}_{\rm s}) \leq \beta(u(\cdot), K_{\rm s})$ is satisfied. It is realized in the form of two procedures: 1. replacement of the admissible control $u(\cdot) \to \overline{u}(\cdot)$ and 2. replacement of the support $K_{\rm s} \to \overline{K}_{\rm s}$. The task of the first phase which lies in constructing the initial support control is solved [5] by the method presented below.

We assume that the following information is known and stored in the computer memory before each iteration: (1) admissible control $u(\cdot)$; (2) support $K_{\rm s} = \{I_{\rm s}, S_{\rm s}\}$; (3) set of nonsupport zeros $S_{\rm ns0}$; (4) matrix $D_{|\rm s|}$; (5) values of $G(t)$ and $\psi_c(t)$, $t \in T_0(j)$, $j \in J$; (6) numbers γ_j, $j \in J$; (7) support values of the pseudocontrol $\omega_j(t)$, $\{j, t\} \in S_{\rm s}$, and qiasicontrol $\widetilde{\omega}_j(t)$, $\{j, t\} \in S_{\rm s}$; (8) vector p; (9) potential vector ν; (10) vector of the pseudo-output ζ and the output $Hx(t^*)$; (11) suboptimality estimate $\beta = \beta(u(\cdot), K_{\rm s})$; and (12) mismatch g.[3]

Prior to proceeding to the iterations, we make sure that for the support control $\{u(\cdot), K_{\rm s}\}$: 1. the principle of ε-maximum (for a given $\varepsilon \geq 0$) is not satisfied and 2. the inequalities (13) and $\|g\| \leq \delta$ are violated.

Replacement of the admissible control. The new admissible control is constructed according to the formula

$$\overline{u}(\cdot) = u(\cdot) + \theta^0 \ell(\cdot), \tag{14}$$

where $\ell(\cdot) = \omega(\cdot) - u(\cdot)$,

$$\theta^0 = \min\{1, \theta_{j_0}(t_0), \theta_{i_0}\}; \quad \theta_{j_0}(t_0) = \min_{\{j,t\} \in S_{\rm s}} \theta_j(t); \ \theta_{i_0} = \min_{i \in I_{\rm ns}} \theta_i;$$

$$\theta_j(t) = \begin{cases} (u_{*j} - u_j(t))/\ell_j(t), & \text{if} \quad \ell_j(t) < 0 \\ (u_j^* - u_j(t))/\ell_j(t), & \text{if} \quad \ell_j(t) > 0 \\ +\infty, & \text{if} \quad \ell_j(t) = 0; \quad \{j, t\} \in S_{\rm s}; \end{cases}$$

[3] Since construction of the elements ν, $\omega_{\rm s}$, $\widetilde{\omega}_{\rm s}$, ζ, and g does without integration, they need not to be stored in the memory, but can be calculated using the corresponding formulas.

$$\theta_i = \begin{cases} (g_{*i} - h'_{(i)}x(t^*))/z_i, & \text{if} \quad z_i < 0 \\ (g_i^* - h'_{(i)}x(t^*))/z_i, & \text{if} \quad z_i > 0 \\ +\infty, & \text{if} \quad z_i = 0; \quad i \in I_{\text{ns}}; \end{cases}$$

$$z = \zeta - Hx(t^*).$$

According to (14), the new control $\overline{u}(\cdot)$ lies on the direction $\ell(\cdot)$ from the control $u(\cdot)$ at the distance θ^0. The admissible direction $\ell(\cdot)$ is that of increasing performance index of problem (1): $\partial c'x(t^*)/\partial\ell(\cdot) = \beta(u(\cdot), K_{\text{s}}) > \varepsilon \geq 0$. The step θ^0 is equal to the maximum step along $\ell(\cdot)$ for which neither direct nor terminal constraints on problem (1) are violated.

The support control $\{\overline{u}(\cdot), K_{\text{s}}\}$ has the suboptimality estimate $\beta(\overline{u}(\cdot), K_{\text{s}}) = (1-\theta^0)\beta(u(\cdot), K_{\text{s}})$. For $\beta(\overline{u}(\cdot), K_{\text{s}}) \leq \varepsilon$, solution of problem (1) is aborted at the ε-optimal control $\overline{u}(t)$, $t \in T_u$. Otherwise, we go to the procedure of support replacement.

Replacement of the support. We distinguish two situations that can arise after the first procedure: (1) $\theta^0 = \theta_{j_0}(t_0)$ and (2) $\theta^0 = \theta_{i_0}$. Let us consider them separately.

(1) Let $\theta^0 = \theta_{j_0}(t_0)$. We begin construction of the new support by calculating the direction $\Delta\nu$ of variation of the potential vector ν: $\Delta\nu_{\text{ns}} = \Delta\nu(I_{\text{ns}}) = 0$; $\Delta\nu_{\text{s}} = \Delta\nu(I_{\text{s}})$ is established from the equation

$$-D'_{\text{s}}\Delta\nu_{\text{s}} = \Delta\delta_{\text{s}} = (\Delta\delta_j(t), \{j,t\} \in S_{\text{s}}),$$

where $\Delta\delta_{j_0}(t_0) = 1$ if $\omega_{j_0}(t_0) > u^*_{j_0}$ and $\Delta\delta_{j_0}(t_0) = -1$ if $\omega_{j_0}(t_0) < u_{*j_0}$, $\Delta\delta_j(t) = 0$, $\{j,t\} \in S_{\text{s}} \setminus \{j_0, t_0\}$.

The initial rate of variation of the performance index of the dual problem of (1) along the direction $\Delta\nu$ [5] is

$$\alpha^1 = -\rho(\omega_{j_0}(t_0), [u_{*j_0}, u^*_{j_0}]) < 0,$$

where $\rho(c, [a,b])$ is the distance from the number c to the interval $[a,b]$.

In terms of the potential vector, replacement of the support is concerned with moving along $\Delta\nu$ until complete relaxation of the piecewise-linear and σ-convex dual performance index at the point $\overline{\nu} = \nu + \sigma^*\Delta\nu$. The "long" dual step σ^* will be calculated in several "short" steps.

To calculate the short steps where the dual performance index undergoes break, we, along with the stored information, make use of the additional information which for the first short step σ^1 has the form: α^1;

$$\sigma_{j_0}(t_0), \tau_{j_0}(t_0), k_0; \quad \sigma_i, i \in I_{\text{s}}; \quad \sigma_j(t_*), \sigma_j(t^*), j \in J; \quad \sigma_j(t), \tau_j(t), \{j,t\} \in S_{\text{ns0}}. \tag{15}$$

To explain the physical implications of numbers (15), we denote by $\Delta\delta(t)$, $t \in T_u$, the variation of co-control generated by the variation of $\Delta\nu$ and determine it from

$$\Delta\delta_j(t) = -\Delta\nu' d_j(t) = -\Delta\nu' \int_t^{t+h} G(\tau)b_j(\tau)d\tau, \quad j \in J, \quad t \in T_u. \tag{16}$$

Each of the numbers $\sigma_j(t)$ and σ_i is the value of the step along $\Delta\nu$ for which either the component of the perturbed co-control $\delta(\sigma, t) = \Delta(t) + \sigma\Delta\delta(t)$, $\sigma \geq 0$, $t \in T_u$, or the component of the perturbed potential vector $\nu(\sigma) = \nu + \sigma\Delta\nu$ vanish, thus increasing the rate of the dual performance index. The numbers $\tau_j(t)$ indicate the direction of the zero of co-control for increasing σ, and k_0 is the index of the instant t_0 in $T_0(j_0)$.

Hence, we get the formulas to calculate numbers (15):

$$\sigma_{j_0}(t_0) = -\Delta_{j_0}(t_0 - h)/\Delta\delta_{j_0}(t_0 - h),\ \tau_{j_0}(t_0) = -1, \quad \text{if} \quad (-1)^{k_0}\gamma_{j_0}\Delta\delta_{j_0}(t_0) > 0;$$

$$\sigma_{j_0}(t_0) = -\Delta_{j_0}(t_0 + h)/\Delta\delta_{j_0}(t_0 + h),\ \tau_{j_0}(t_0) = 1, \quad \text{if} \quad (-1)^{k_0}\gamma_{j_0}\Delta\delta_{j_0}(t_0) < 0;$$

$$\sigma_i = \begin{cases} -\nu_i/\Delta\nu_i, & \text{if} \quad \nu_i\Delta\nu_i < 0 \\ \infty, & \text{if} \quad \nu_i\Delta\nu_i \geq 0; \quad i \in I_{\text{s}}; \end{cases}$$

$$\sigma_j(t_*) = \begin{cases} -\Delta_j(t_*)/\Delta\delta_j(t_*), & \text{if} \quad \Delta_j(t_*)\Delta\delta_j(t_*) < 0 \\ \infty, & \text{if} \quad \Delta_j(t_*)\Delta\delta_j(t_*) \geq 0; \quad j \in J; \end{cases}$$

$$\sigma_j(t^*) = \begin{cases} -\Delta_j(t^* - h)/\Delta\delta_j(t^* - h), & \text{if } \Delta_j(t^* - h)\Delta\delta_j(t^* - h) < 0 \\ \infty, & \text{if } \Delta_j(t^* - h)\Delta\delta_j(t^* - h) \geq 0;\ j \in J; \end{cases}$$

$$\sigma_j(t) = -\Delta_j(t - h)/\Delta\delta_j(t - h),\ \tau_j(t) = -1,\ \text{if } \Delta_j(t - h)\Delta\delta_j(t - h) < 0;$$

$$\sigma_j(t) = -\Delta_j(t)/\Delta\delta_j(t),\ \tau_j(t) = 1, \quad \text{if } \Delta_j(t)\Delta\delta_j(t) < 0,\ \{j, t\} \in S_{\text{ns0}}.$$

Here,

$$\Delta_j(t) = \int\limits_t^{t+h} (\psi_c'(\tau) - \nu' G(\tau))b_j(\tau)d\tau = \int\limits_t^{t+h} \psi_c'(\tau)b_j(\tau)d\tau + \nu' d_j(t);$$

$$\Delta\delta_j(t) = -\int\limits_t^{t+h} \Delta\nu' G(\tau)b_j(\tau)d\tau = -\Delta\nu' d_j(t);\ d_j(t) = \int\limits_t^{t+h} G(\tau)b_j(\tau)d\tau;$$

$$\Delta_j(t + h) = \int\limits_{t+h}^{t+2h} \psi_c'(\tau)b_j(\tau)d\tau + \nu' d_j(t + h);\ \Delta\delta_j(t + h) = -\Delta\nu' d_j(t + h);$$

$$d_j(t + h) = \int\limits_{t+h}^{t+2h} G(\tau)b_j(\tau)d\tau;$$

$$\Delta_j(t - h) = \int\limits_{t-h}^{t} \psi_c'(\tau)b_j(\tau)d\tau + \nu' d_j(t - h);\ \Delta\delta_j(t - h) = -\Delta\nu' d_j(t - h);$$

$$d_j(t - h) = \int\limits_{t-h}^{t} G(\tau)b_j(\tau)d\tau.$$

To calculate numbers (15), it suffices, therefore, to take the stored values as the initial states $\psi_c(t)$ and $G(t)$ and integrate Eqs. (7) and (8) over the interval $[t, t + 2h]$ or $[t - h, t]$.

Prior to performing the short steps, we perform the "zero" step to transform the stored information under the assumption that an infinitely small step $\sigma > 0$ has been made in the direction $\Delta\nu$.

(1) For $t_* < t_0 < t^* - h$, $\tau_{j_0}(t_0) = 1$, we assume that $p^1 = p + d_{j_0}(t_0)\omega_{j_0}^{k_0 - 1}$; $S_{\text{ns0}}^1 = S_{\text{ns0}} \bigcup \{j_0, t_0 + h\}$; and store $G(t_0 + h)$, $\psi_c(t_0 + h)$ instead of $G(t_0)$, $\psi_c(t_0)$.

(2) If $t_* < t_0 \leq t^* - h$, $\tau_{j_0}(t_0) = -1$, then $p^1 = p + d_{j_0}(t_0)\omega_{j_0}^{k_0}$; $S_{\text{ns0}}^1 = S_{\text{ns0}} \bigcup \{j_0, t_0\}$.

(3) For $t_0 = t_*$, $\tau_{j_0}(t_0) = 1$, we assume that $\gamma_{j_0}^1 = -\gamma_{j_0}$; $p^1 = p + d_{j_0}(t_*)u_{*j_0}$ if $\gamma_{j_0}^1 = -1$; $p^1 = p + d_{j_0}(t_*)u_{j_0}^*$ if $\gamma_{j_0}^1 = 1$; $S_{\text{ns0}}^1 = S_{\text{ns0}} \bigcup \{j_0, t_* + h\}$; $k^1(j_0) = k(j_0) + 1$; the points of the set $T_0(j_0)$ are re-enumerated.

(4) For $t_0 = t_*, \tau_{j_0}(t_0) = -1$, we assume that $p^1 = p + d_{j_0}(t_*)\omega_{j_0}^{k_0}$ and $S^1_{\rm ns0} = S_{\rm ns0}$. This case is treated as the disappearance of the support zero $\{j_0, t_*\}$ through the left boundary of the set T_u.

(5) If $t_0 = t^* - h, \tau_{j_0}(t_0) = 1$, then we assume that $k^1(j_0) = k(j_0) - 1$, $p^1 = p + d_{j_0}(t^* - h)\omega_{j_0}^{k_0-1}$, and $S^1_{\rm ns0} = S_{\rm ns0}$ and delete the values of $G(t^* - h), \psi_c(t^* - h)$. This case is treated as disappearance of the support zero $\{j_0, t^* - h\}$ through the right boundary of T_u.

Let us assume that the $(\ell-1)$th short step $\sigma^1, \dots, \sigma^{\ell-1}$ was calculated along the direction $\Delta\nu$ and that before the ℓth short step the information was as follows: (1) the sets $S^\ell_{\rm ns0}$; $S^\ell = S^\ell_{\rm ns0} \bigcup S_* \bigcup S^*$, $S_* = \bigcup\limits_{j\in J}\{j, t_*\}$, $S^* = \bigcup\limits_{j\in J}\{j, t^*\}$; (2) the numbers $\sigma^\ell_j(t)$, $\{j,t\} \in S^\ell$, σ^ℓ_i, $i \in I_{\rm s}$; (3) the numbers γ^ℓ_j, $j \in J$, and $\omega_j^{k,\ell}$, $k \in K^\ell(j) = \{0, 1, \dots, k^\ell(j)\}$,[4] $j \in J$; (4) the vector p^ℓ; (5) the values of $G(t)$ and $\psi_c(t)$, $t \in T^\ell_{\rm ns0}(j) \bigcup t_0$, $j \in J$; (6) the suboptimality estimate β^ℓ; and (7) the rate of variation of the dual performance index α^ℓ.

To simplify calculations, we assume that all numbers $\sigma^\ell_j(t)$, $\{j,t\} \in S^\ell$; σ_i, $i \in I_{\rm s}$, are different, with the possible exception of the pair $\sigma^\ell_j(t)$, $\sigma^\ell_j(t+h)$ for some $t \in T_u \setminus t^*$, $j \in J$. The case of coincidence of other numbers (15) was considered in [5].

On the basis of information 1–7, we calculate the ℓth short step

$$\sigma^\ell = \min\{\sigma_{j^\ell}(t^\ell), \sigma_{i^\ell}\}, \tag{17}$$

where

$$\sigma_{j^\ell}(t^\ell) = \min_{\{j,t\}\in S^\ell} \sigma_j(t); \quad \sigma_{i^\ell} = \min_{i\in I_{\rm s}} \sigma_i,$$

which allows one to indicate the interval $[\sigma^{\ell-1}, \sigma^\ell[$ where the dual performance index decreases with the rate α^ℓ and calculate its suboptimality estimate

$$\beta^{\ell+1} = \beta^\ell + \alpha^\ell(\sigma^\ell - \sigma^{\ell-1}),\ \beta^1 = \beta(\overline{u}(\cdot), K_{\rm s}).$$

If $\beta^{\ell+1} \le \varepsilon$, then the solution of problem (1) is aborted at the ε-optimal control $\overline{u}(\cdot)$. Otherwise, we calculate the rate increment of the dual performance index at the point σ^ℓ for the motion along $\Delta\nu$:

$$\Delta\alpha^\ell = \begin{cases} (u^*_{j^\ell} - u_{*j^\ell})|\Delta\delta_{j^\ell}(t^\ell)|, & \text{if} \quad \sigma^\ell = \sigma_{j^\ell}(t^\ell) \\ (g^*_{i^\ell} - g_{*i^\ell})|\Delta\nu_{i^\ell}|, & \text{if} \quad \sigma^\ell = \sigma_{i^\ell}. \end{cases}$$

The rate of variation of the dual performance index in the right neighborhood of the point σ^ℓ is $\alpha^{\ell+1} = \alpha^\ell + \Delta\alpha^\ell$. If the inequality

$$\alpha^{\ell+1} \ge 0 \tag{18}$$

is satisfied, then iteration is completed, that is, it is assumed that $\sigma^* = \sigma^\ell$, and the information for a new iteration (see below) is generated. If inequality (18) is not satisfied, then one proceeds to generating information 1–7 for the next, $(\ell+1)$th step.

We distinguish two possibilities: (a) $\sigma^\ell = \sigma_{j^\ell}(t^\ell)$ and (b) $\sigma^\ell = \sigma_{i^\ell}$.

(a) The stored information is transformed depending on the following situations.

A. Let $\sigma_{j^\ell}(t^\ell)$ be a unique number satisfying (17) and k^ℓ be the index of the instant t^ℓ in $T_0(j^\ell)$.

A.1. If $\{j^\ell, t^\ell\} \in S^\ell_{\rm ns0}$ and $\tau_{j^\ell}(t^\ell) = 1$, then we assume that $p^{\ell+1} = p^\ell - d_{j^\ell}(t^\ell)(u^*_{j^\ell} - u_{*j^\ell})(-1)^{k^\ell}\gamma^\ell_{j^\ell}$ and $S^{\ell+1}_{\rm ns0} = (S^\ell_{\rm ns0} \setminus \{j^\ell, t^\ell\}) \bigcup \{j^\ell, t^\ell + h\}$ and calculate the new step $\sigma_{j^\ell}(t^\ell + h) = -\Delta_{j^\ell}(t^\ell + h)/\Delta\delta_{j^\ell}(t^\ell + h)$. Instead of $G(t^\ell)$ and $\psi_c(t^\ell)$, we store $G(t^\ell + h)$ and $\psi_c(t^\ell + h)$.

[4] The numbers $\omega_j^{k,\ell}$ need not to be stored because they can be easily calculated from (12).

A.2. For $\{j^\ell, t^\ell\} \in S^\ell_{\rm ns0}$, $\tau_{j^\ell}(t^\ell) = -1$, we assume that $p^{\ell+1} = p^\ell + d_{j^\ell}(t^\ell - h)(u^*_{j^\ell} - u_{*j^\ell})(-1)^{k^\ell}\gamma^\ell_{j^\ell}$, $S^{\ell+1}_{\rm ns0} = (S^\ell_{\rm ns0} \setminus \{j^\ell, t^\ell\}) \bigcup \{j^\ell, t^\ell - h\}$, and $\sigma_{j^\ell}(t^\ell - h) = -\Delta_{j^\ell}(t^\ell - 2h)/\Delta\delta_{j^\ell}(t^\ell - 2h)$. Instead of $G(t^\ell)$ and $\psi_c(t^\ell)$, we store $G(t^\ell - h)$ and $\psi_c(t^\ell - h)$.

A.3. If $\{j^\ell, t^\ell\} \in S_*$, then we enter into $S^\ell_{\rm ns0}$ a new nonsupport zero $\{j^\ell, t^\ell + h\}$: $S^{\ell+1}_{\rm ns0} = S^\ell_{\rm ns0} \bigcup \{j^\ell, t^\ell + h\}$. Then, $k^{\ell+1}(j^\ell) = k^\ell(j^\ell) + 1$, the points of the set $T^{\ell+1}_{\rm s}(j^\ell)$ are re-enumerated and it is assumed that $p^{\ell+1} = p^\ell - d_{j^\ell}(t^\ell)(u^*_{j^\ell} - u_{*j^\ell})\gamma^\ell_{j^\ell}$; $\gamma^{\ell+1}_{j^\ell} = -\gamma^\ell_{j^\ell}$; $\tau_{j^\ell}(t^\ell) = 1$. We calculate a new step $\sigma_{j^\ell}(t^\ell + h) = -\Delta_{j^\ell}(t^\ell + h)/\Delta\delta_{j^\ell}(t^\ell + h)$. This situation is treated as the appearance of a new zero of the function $\delta(\sigma, t)$, $\sigma \geq 0$, $t \in T_u$, at the left end of T_u.

A.4. For $\{j^\ell, t^\ell\} \in S^*$, we introduce into $S^\ell_{\rm ns0}$ the new pair $\{j^\ell, t^\ell - h\}$: $S^{\ell+1}_{\rm ns0} = S^\ell_{\rm ns0} \bigcup \{j^\ell, t^\ell - h\}$ and assume that $k^{\ell+1}(j^\ell) = k^\ell(j^\ell) + 1$; $p^{\ell+1} = p^\ell - d_{j^\ell}(t^\ell - h)(u^*_{j^\ell} - u_{*j^\ell})(-1)^{k^\ell(j^\ell)}\gamma^\ell_{j^\ell}$; $\tau_{j^\ell}(t^\ell) = -1$. We re-enumerate the points of the set $T_0(j^\ell)$; store the values of $G(t^\ell - h), \psi_c(t^\ell - h)$; and determine $\sigma_{j^\ell}(t^\ell - h) = -\Delta_{j^\ell}(t^\ell - 2h)/\Delta\delta_{j^\ell}(t^\ell - 2h)$. The situation is treated as the appearance of the zero of $\delta(\sigma, t)$, $\sigma \geq 0$, $t \in T_u$, at the point $t^\ell - h$.

B. Let the step σ^ℓ in (17) be attained on two pairs, $\{j^\ell, t^\ell\}$ and $\{j^\ell, t^\ell + h\}$: $\sigma_{j^\ell}(t^\ell) = \sigma_{j^\ell}(t^\ell + h)$.

B.1. In the case of $\{j^\ell, t^\ell\} \in S_*$, we remove from the set $S^\ell_{\rm ns0}$ the point $\{j^\ell, t^\ell + h\}$: $S^{\ell+1}_{\rm ns0} = S^\ell_{\rm ns0} \setminus \{j^\ell, t^\ell + h\}$, assume that $p^{\ell+1} = p^\ell + d_{j^\ell}(t^\ell)(u^*_{j^\ell} - u_{*j^\ell})(-1)^{k^\ell}\gamma^\ell_{j^\ell}$; $\gamma^{\ell+1}_{j^\ell} = -\gamma^\ell_{j^\ell}$; $\sigma_{j^\ell}(t^\ell) = \infty$; $k^{\ell+1}(j^\ell) = k^\ell(j^\ell) - 1$, re-enumerate the points of the set $T_0(j^\ell)$, and erase the values $G(t^\ell + h)$, $\psi_c(t^\ell + h)$. Here, the zero $\{j^\ell, t^\ell + h\}$ is moved to the point t_* if $\{j^\ell, t_*\} \notin S_{\rm s}$ or $t_* + h$ if $\{j^\ell, t_*\} \in S_{\rm s}$, and disappears for $\sigma > \sigma^\ell$ through the left boundary of T_u.

B.2. For $\{j^\ell, t^\ell + h\} \in S^*$, we remove from the set $S^\ell_{\rm ns0}$ the pair $\{j^\ell, t^\ell\}$: $S^{\ell+1}_{\rm ns0} = S^\ell_{\rm ns0} \setminus \{j^\ell, t^\ell\}$, $k^{\ell+1}(j^\ell) = k^\ell(j^\ell) - 1$; erase the values of $G(t^\ell), \psi_c(t^\ell)$; re-enumerate the points $T^{\ell+1}_{\rm s}(j^\ell)$; and assume that $p^{\ell+1} = p^\ell - d_{j^\ell}(t^\ell)(u^*_{j^\ell} - u_{*j^\ell})(-1)^{k^\ell}\gamma^\ell_{j^\ell}$ and $\sigma_{j^\ell}(t^\ell + h) = \infty$. The zero $\{j^\ell, t^\ell\}$ disappears through the right boundary of the interval T_u for $\sigma > \sigma^\ell$.

B.3. If $\{j^\ell, t^\ell\}$, $\{j^\ell, t^\ell + h\} \in S^\ell_{\rm ns0}$, then we assume that $p^{\ell+1} = p^\ell - d_{j^\ell}(t^\ell)(u^*_{j^\ell} - u_{*j^\ell}) \times (-1)^{k^\ell}\gamma^\ell_{j^\ell}$, $S^{\ell+1}_{\rm ns0} = S^\ell_{\rm ns0} \setminus \{\{j^\ell, t^\ell\}, \{j^\ell, t^\ell + h\}\}$, and $k^{\ell+1}(j^\ell) = k^\ell(j^\ell) - 2$; purge the values of $G(t^\ell)$, $G(t^\ell + h)$, $\psi_c(t^\ell)$, and $\psi_c(t^\ell + h)$; and re-enumerate the points of $T^{\ell+1}_{\rm s}(j^\ell)$. This situation is treated as conglutination and disappearance of two zeros at the point t^ℓ for $\sigma > \sigma^\ell$.

C. If Case A was realized and $\{j^\ell, t^\ell + h\} \in S_{\rm s}$, $\tau_{j^\ell}(t^\ell) = 1$, then we assume that $p^{\ell+1} = p^\ell - d_{j^\ell}(t^\ell)(u^*_{j^\ell} - u_{*j^\ell})(-1)^{k^\ell}\gamma^\ell_{j^\ell}$; $S^{\ell+1}_{\rm ns0} = (S^\ell_{\rm ns0} \setminus \{j^\ell, t^\ell\}) \bigcup \{j^\ell, t^\ell + 2h\}$ and calculate the new step $\sigma_{j^\ell}(t^\ell + 2h) = -\Delta_{j^\ell}(t^\ell + 2h)\ /\ \Delta\delta_{j^\ell}(t^\ell + 2h)$. Instead of $G(t^\ell)$ and $\psi_c(t^\ell)$, we store $G(t^\ell + 2h)$ and $\psi_c(t^\ell + 2h)$. For $\{j^\ell, t^\ell - 2h\} \in S_{\rm s}$, $\tau_{j^\ell}(t^\ell) = -1$, we assume that $S^{\ell+1}_{\rm ns0} = (S^\ell_{\rm ns0} \setminus \{j^\ell, t^\ell\}) \bigcup \{j^\ell, t^\ell - 2h\}$; $\sigma_{j^\ell}(t^\ell - 2h) = -\Delta_{j^\ell}(t^\ell - 3h)\ /\ \Delta\delta_{j^\ell}(t^\ell - 3h)$, $p^{\ell+1} = p^\ell + d_{j^\ell}(t^\ell)(u^*_{j^\ell} - u_{*j^\ell})(-1)^{k^\ell}\gamma^\ell_{j^\ell}$. We re-enumerate the elements of $T_0(j^\ell)$. This situation is treated as the passage of the mobile zero through the support one.

The rest of the information for the $(\ell + 1)$th step is set down without changes.

For the new iteration, the information is transformed as follows:

A.1. We assume that $\overline{p} = p^\ell - d_{j^\ell}(t^\ell)\omega^{k^\ell,\ell}_{j^\ell}$; $\overline{S}_{\rm s} = (S_{\rm s} \setminus \{j_0, t_0\}) \bigcup \{j^\ell, t^\ell\}$; $\overline{S}_{\rm ns0} = S^\ell_{\rm ns0} \setminus \{j^\ell, t^\ell\}$.

A.2. For $\tau_{j^\ell}(t^\ell) = -1$, we assume that $\overline{p} = p^\ell - d_{j^\ell}(t^\ell - h)\omega^{k^\ell - 1,\ell}_{j^\ell}$; $\overline{S}_{\rm s} = (S_{\rm s} \setminus \{j_0, t_0\}) \bigcup \{j^\ell, t^\ell - h\}$; $\overline{S}_{\rm ns0} = S^\ell_{\rm ns0} \setminus \{j^\ell, t^\ell\}$ and store $G(t^\ell - h)$ and $\psi_c(t^\ell - h)$ instead of $G(t^\ell)$ and $\psi_c(t^\ell)$.

A.3. We assume that $\overline{p} = p^\ell - d_{j^\ell}(t^\ell)\omega^{k^\ell,\ell}_{j^\ell}$; $\overline{S}_{\rm s} = (S_{\rm s} \setminus \{j_0, t_0\}) \bigcup \{j^\ell, t^\ell\}$; $\overline{S}_{\rm ns0} = S^\ell_{\rm ns0}$.

A.4. In this case, we assume that $\overline{p} = p^\ell - d_{j^\ell}(t^\ell - h)\omega^{k^\ell - 1,\ell}_{j^\ell}$; $\overline{S}_{\rm s} = (S_{\rm s} \setminus \{j_0, t_0\}) \bigcup \{j^\ell, t^\ell - h\}$; $\overline{S}_{\rm ns0} = S^\ell_{\rm ns0}$; $\overline{k}(j^\ell) = k^\ell(j^\ell) + 1$, re-enumerate the points of the set $T_0(j^\ell)$, and store the values of $G(t^\ell - h), \psi_c(t^\ell - h)$.

B.1. We remove the pair $\{j^\ell, t^\ell+h\}$ from the set $S^\ell_{\rm ns0}$: $\overline{S}_{\rm ns0} = S^\ell_{\rm ns0}\setminus\{j^\ell, t^\ell+h\}$; $\overline{k}(j^\ell) = k^\ell(j^\ell)-1$, erase the values of $G(t^\ell+h)$, $\psi_c(t^\ell+h)$, and re-enumerate the points of the set $T_0(j^\ell)$. We assume that $\overline{p} = p^\ell - d_{j^\ell}(t^\ell-h)\omega^{k^\ell-1,\ell}_{j^\ell}$; $\overline{\gamma}_{j^\ell} = -\gamma^\ell_{j^\ell}$; $\overline{S}_{\rm s} = (S_{\rm s}\setminus\{j_0,t_0\})\bigcup\{j^\ell,t^\ell\}$.

B.2. In this case, we assume that $\overline{p} = p^\ell - d_{j^\ell}(t^\ell)\omega^{k^\ell,\ell}_{j^\ell}$; $\overline{S}_{\rm s} = (S_{\rm s}\setminus\{j_0,t_0\})\bigcup\{j^\ell,t^\ell\}$; $\overline{S}_{\rm ns0} = S^\ell_{\rm ns0}\setminus\{j^\ell,t^\ell\}$.

The new support has the form $\overline{K}_{\rm s} = (I_{\rm s}, \overline{S}_{\rm s})$. To obtain $\overline{D}_{|{\rm s}|}$, in the matrix $D_{|{\rm s}|}$ we replace the column $d_{j_0}(t_0)$ by the column $d_{j^\ell}(t^\ell-h)$ (in Cases A.2 and A.4) or $d_{j^\ell}(t^\ell)$ (in the remaining cases).

Note 1. If $\{j,t_*\}\in S_{\rm s}$, then in S_* the pair $\{j,t_*\}$ is replaced by $\{j,t_*+h\}$ and $\sigma_j(t_*+h) = -\Delta_{j^\ell}(t_*+h)/\Delta\delta_{j^\ell}(t_*+h)$ is calculated. Similarly, if $\{j,t^*-h\}\in S_{\rm s}$, we assume that $S^* := (S^*\setminus\{j,t^*\})\bigcup\{j,t^*-h\}$ and calculate $\sigma_j(t^*-h) = -\Delta_{j^\ell}(t^*-2h)/\Delta\delta_{j^\ell}(t^*-2h)$.

Note 2. We disregarded above for simplicity the case where, upon moving along the direction $\Delta\nu$, new zeros arise inside the set T_u. To study it, one needs to complement the information of (15) by the totalities of the pairs $\{j,t\}$ and the numbers $\sigma_j(t)$. The former totality consists of the stationary points of the varied co-control $\delta(\sigma,t)>0$, $\sigma>0$, $t\in T_u$, (the point $t\in T_u$ is stationary if for some $\sigma>0$ either $\delta_j(\sigma,t)>0$, $\delta_j(\sigma,t-h)>\delta_j(\sigma,t)$, $\delta_j(\sigma,t+h)>\delta_j(\sigma,t)$ or $\delta_j(\sigma,t)<0$, $\delta_j(\sigma,t-h)<\delta_j(\sigma,t)$, $\delta_j(\sigma,t+h)<\delta_j(\sigma,t)$) is satisfied. The numbers $\sigma_j(t)$ characterize the value of steps along $\Delta\nu$ for which the jth component of the varied co-control vanishes at time t: $\delta_j(\sigma_j(t),t)=0$. By following the motion and position of the stationary points of the varied co-control, one can detect appearance of a new "internal" zero of the co-control. In more complicated situations, understandably, new stationary points can occur that are detected by means of the stationary points of the first derivative of the co-control. The amount of additional information which depends on the complexity of a particular problem (1) can be established for the regular problems (1) in each particular case. Additional information is handled similar to the above stationary case.

Now we consider Case (b) where $\sigma^\ell = \sigma_{i^\ell}$. If inequality (18) is satisfied, then we pass to a new iteration with $\overline{K}_{\rm s} = \{\overline{I}_{\rm s}, \overline{S}_{\rm s}\}$, $\overline{I}_{\rm s} = I_{\rm s}\setminus i^\ell$, $\overline{S}_{\rm s} = S_{\rm s}\setminus\{j_0,t_0\}$; $\overline{S}_{\rm ns0} = S^\ell_{\rm ns0}$; $\overline{\gamma}_j = \gamma^\ell_j$, $j\in J$; $\overline{p}_k(j) = p^\ell_k(j)$, $k\in\overline{K}(j) = K^\ell(j)$; $G(t)$, $\psi_c(t)$, $t\in\overline{T}_{\rm ns0}(j)$, $j\in J$; $\beta(\overline{u},\overline{K}_{\rm s}) = \beta^\ell$. The matrix $\overline{D}_{|{\rm s}|}$ is obtained by removing the column $d_{j_0}(t_0)$ from $D_{|{\rm s}|}$. Otherwise ($\alpha^{\ell+1}<0$), we assume that $\sigma_{i^\ell}=\infty$ and go to the $(\ell+1)$th step.

According to [5], there necessarily should be an ℓ_0 such that $\alpha^{\ell_0}<0$ and $\alpha^{\ell_0+1}\geq 0$.

(2) Let us consider the situation where the first procedure realized the equality $\theta^0 = \theta_{i_0}$. The variation of the potential vector $\Delta\nu$ is constructed as follows: $\Delta\nu_{i_0}=1$ if $\zeta_{i_0}>g^*_{i_0}$; $\Delta\nu_{i_0}=-1$ if $\zeta_{i_0}<g_{*i_0}$; $\Delta\nu_i=0$, $i\in I_{\rm ns}\setminus i_0$; $\Delta\nu_{\rm s}$ is established from the equation

$$-D'_{\rm s}\Delta\nu_{\rm s} = (d_{i_0 j}(t), \{j,t\}\in S_{\rm s})\Delta\nu_{i_0}.$$

Variation of the co-control obeys formulas (16). The initial rate of variation of the dual performance index along $\Delta\nu$ is now as follows [5]:

$$\alpha^1 = -\rho(\zeta_{i_0}, [g_{*i_0}, g^*_{i_0}]) < 0.$$

The operations for calculation of the long step σ^* in this case are similar to the operations for situation 1, the only difference being the lack of the "zero" step. The new support is constructed according to the rules

(a) $(\sigma^* = \sigma_{j^{\ell_0}}(t^{\ell_0}))$ $\quad\overline{K}_{\rm s} = \{\overline{I}_{\rm s}, \overline{S}_{\rm s}\}$, $\overline{I}_{\rm s} = I_{\rm s}\bigcup i_0$. In cases A.2 and A.4, $\overline{S}_{\rm s} = S_{\rm s}\bigcup\{j^\ell, t^\ell-h\}$; we add a new column $d_{j^\ell}(t^\ell-h)$ to the matrix $D_{|{\rm s}|}$. In the remaining cases, we assume that

$S_{\rm s} = S_{\rm s} \bigcup \{j^\ell, t^\ell\}$ and add the column $d_{j^\ell}(t^\ell)$ to $D_{|{\rm s}|}$. The rules for constructing the set $\overline{S}_{\rm ns0}$ are similar to Case 1a.

(b) $(\sigma^* = \sigma_{i^{\ell_0}})$ $\quad \overline{K}_{\rm s} = \{\overline{I}_{\rm s}, S_{\rm s}\}$, $\overline{I}_{\rm s} = (I_{\rm s} \setminus i^{\ell_0}) \bigcup i_0$; $\overline{S}_{\rm ns0} = S^{\ell_0}_{\rm ns0}$; $\overline{D}_{|{\rm s}|} = D_{|{\rm s}|}$.

On efficiency of the method. The number of full (over the entire interval T) integrations of the direct or adjoint system that are required for optimal (suboptimal) control [2] is a natural characteristic of efficiency of the methods for solution of the problem of optimal control. If several direct and adjoint systems are integrated simultaneously and independently to solve the problem, then, with regard for parallelization of calculations, these integrations can be regarded as a single integration. In this connection, one integration will be used as a unit of laboriousness of the method. As was shown in Section 4, the procedures for identification of the supports and suboptimal controls feature such laboriousness. Therefore, the laboriousness of preparing the first iteration under the known initial admissible control and nonempty support is equal to two. The laboriousness of preparing the additional information at the kth iteration is $2(\,|S^1_{\rm ns0}(k)| + r + 1\,)/N$. Let at the kth iteration $L(k)$ be the number of short steps where the dual step $\sigma^\ell = \sigma_{j^\ell}(t^\ell)$ was realized. Laboriousness of each short step is $2/N$ if Situations A and C are realized and $1/N$ for Situation B and the last step of the iteration. Let $P(k)$ be the number of occurrences of Situation B at the kth iteration.[5] Then, the laboriousness of iteration is $(2L(k) - P(k))/N$, and the total laboriousness of the method follows the formula

$$E = 2 + \sum_{k=1}^{k_*} \frac{2(\,|S^1_{\rm ns0}(k)| + r + 1\,) + 2L(k) - P(k)}{N},$$

where k_* is the number of iterations required to construct the ε-optimal control. Laboriousness E can be dramatically reduced if for each element of $S^1(k)$ the additional information is calculated by $|S^1(k)|$ parallel processors and not successively. Then, laboriousness of preparing the additional information is $2/N$ and $E = 2 + \sum\limits_{k=1}^{k_*} (2(L(k) + 1) - P(k))/N$. Analytic representation of E in terms of the parameters of problem (1) is a challenge. One can get some idea of efficiency of the method from the results of numerical solution of the nontrivial example in Section 9. Experimental evaluation of the efficiency of the method and its comparison with other methods is outside the scope of the present paper.

7. DUAL METHOD

The dual method is pivotal to designing the optimal systems. It does without the information about the initial admissible control. We construct it on the basis of the procedure for replacing the support of the direct method for solving problem (1) described in Section 6.

Let some support $K_{\rm s} = \{I_{\rm s}, S_{\rm s}\}$ be known. An empty support can be used as the initial one. We use the support $K_{\rm s}$ to construct the accompanying pseudocontrol $\omega(t)$, $t \in T_u$, pseudo-output ζ, qiasicontrol $\widetilde{\omega}(t)$, $t \in T_u$, and the mismatch vector g (Section 4). If the pseudocontrol and pseudo-output satisfy inequalities (13), then $u^0(t) = \omega(t)$, $t \in T_u$, is the optimal control. If the inequality $\|g\| \le \delta$ is satisfied for the given $\delta \ge 0$, then the qiasicontrol $\widetilde{\omega}(t)$, $t \in T_u$, is the 0δ-solution of problem (1).

Let us assume that $\varepsilon\delta$-solution was not detected. We assume that $\rho_i = \rho(\zeta_i, [g_{*i}, g^*_i])$, $i \in I$, and $\rho_j(t) = \rho(\omega_j(t), [u_{*j}, u^*_j])$, $t \in T_u$, $j \in J$, and calculate

$$\rho_0 = \max\{\rho_{j_0}(t_0), \rho_{i_0}\},\ \ \rho_{j_0}(t_0) = \max_{\{j,t\} \in S_{\rm s}} \rho_j(t),\ \ \rho_{i_0} = \max_{i \in I_{\rm ns}} \rho_i. \tag{19}$$

[5] The number $P(k)$ includes also the last short step of iteration.

Iteration of the dual method of solution of (1) lies in replacing the support of the direct method (Section 6) where the elements determined from (19) are used as $\{j_0, t_0\}$, i_0. For the condition of nondegeneracy, the method is finite. A modification of the dual method that is finite for any problem of linear programming was described in [5].

8. DESIGN OF THE OPTIMAL FEEDBACK CONTROLS

The notion of optimal feedback control was introduced in Section 2. As was noted in the Introduction, the problem of optimal design is pivotal to the control theory. In one's time, great hopes were pinned on the dynamic programming [11]. In contrast to the principle of maximum that is oriented to the program solutions, the dynamic programming is aimed at constructing positional solutions. The analytical difficulties that were regarded as the main obstacles to using the dynamic programming in the optimal control theory are eliminated for the formulation of the problem that was considered in this paper. The Bellman equations in the class of discrete controls are recurrent and strictly substantiated.[6] However, the famous Bellmans's "curse of dimensionality" prevents efficient design of the optimal feedback controls even for more or less serious linear problems.

A new approach to designing the optimal systems was proposed in [8, 12]. It lies in abandoning the intent to solve the problem in analytic (formal) terms and suggesting a constructive realization of the optimal feedback based on the analysis of its use in particular control processes. The feedbacks are known to be introduced in the control theory to offset the perturbations disregarded by the mathematical model (1). We assume that the optimal feedback $u^0(t,x)$, $x \in X_\tau$, $t \in T_u$, has been constructed and consider behavior of the closed system under permanent perturbations

$$\dot{x} = A(t)x + B(t)u^0(t,x) + w(t), \quad x(t_*) = x_0. \tag{20}$$

Let a previously unknown sectionally continuous perturbation $w^*(t)$, $t \in T$, be realized in some particular control process. It generates a particular trajectory $x^*(t)$, $t \in T$, of the closed system (20) satisfying the identity

$$\dot{x}^*(t) \equiv A(t)x^*(t) + B(t)u^0(t, x^*(t)) + w^*(t), \quad x^*(t_*) = x_0,$$

from which it is evident that the control does without the entire optimal feedback (for all $x \in X_\tau$, $\tau \in T_u$) and makes use only of its values $u^*(t) = u^0(t, x^*(t))$, $t \in T_u$, along the isolated curve $x^*(t)$, $t \in T$. We call the function $u^*(t)$, $t \in T_u$, the realization of the optimal feedback in a particular control process. A device capable of calculating for each $\tau \in T_u$ the value of $u^*(\tau)$ for the current position $(\tau, x^*(\tau))$ over the time $s(\tau)$ that does not exceed h will be called the optimal controller realizing the optimal realtime feedback.[7]

The algorithm of the optimal controller relies on the definition of the optimal feedback (3) and the dual method of Section 7. Let us assume that the algorithm was constructed and, by operating at the instants t_*, $t_* + h, \ldots, \tau$, the optimal controller generated the controls $u^*(t_*)$, $u^*(t_* + h), \ldots, u^*(\tau)$. Let under the action of these controls and the realized perturbation $w^*(t)$, $t \in [t_*, \tau + h]$, the control system be at the current time $\tau + h$ in the state $x^*(\tau + h)$. According to definition (3), to generate the control $u^*(\tau + h)$, one needs the solution $u^0(t \,|\, \tau + h, x^*(\tau + h))$, $t \in T(\tau{+}h)$, of problem (2) for the position $(\tau{+}h, x^*(\tau{+}h))$. By assumption, at the previous instant τ the optimal controller already solved problem (2) for the position $(\tau, x^*(\tau))$. Under the limited perturbation $w^*(t)$, $t \in [\tau, \tau + h[$, and sufficiently small h, the state $x^0(\tau + h)$ into which the control system would pass with $w^*(t) = 0$, $t \in [\tau, \tau + h[$, from the position $(\tau, x^*(\tau))$ differs insignificantly

[6] The Bellman equation recently was strictly substantiated within the framework of nonsmooth analysis and in the class of measurable controls.

[7] We disregard here the delay $s(\tau)$ which often is very small and has almost no impact on the trajectory $x^*(t)$, $t \in T$.

from the actual state $x^*(\tau+h)$, which allows one to make use of the fundamental property of dual methods that is very useful for designing the optimal systems: the solution $u^0(t\,|\,\tau+h, x^*(\tau+h))$, $t \in T(\tau+h)$, of problem (2) for $\tau+h$ is rapidly constructed from the optimal support $K_{\rm s}^0(\tau+h)$ obtained by correcting the optimal support $K_{\rm s}^0(\tau)$ of problem (2) for τ that is close to the preceding problem.

The functional form of problem (2) which the optimal controller solved at time τ is as follows:

$$\begin{gathered}\sum_{t\in T_u(\tau)} c'(t)u(t) \to \max,\\ g_*(\tau) \le \sum_{t\in T_u(\tau)} D(t)u(t) \le g^*(\tau),\\ u_* \le u(t) \le u^*, t \in T_u(\tau) = \{\tau, \tau+h, \dots, t^*-h\},\end{gathered} \tag{21}$$

where $g_*(\tau) = g_* - G(\tau)x^*(\tau)$, $g^*(\tau) = g^* - G(\tau)x^*(\tau)$.

Solution of problem (21) gave rise to the following stored information: (1) support $K_{\rm s}^0(\tau) = \{I_{\rm s}^0(\tau), S_{\rm s}^0(\tau)\}$; (2) set of nonsupport zeros $S_{\rm ns0}(\tau)$; (3) matrices $D_{|{\rm s}|}(\tau)$ and $D(\tau)$; (4) values of $G(t)$ and $\psi_c(t)$, $t \in T_{\rm s}(j\,|\,\tau)$, $j \in J$; (5) numbers $\gamma_j(\tau)$, $j \in J$; (6) pseudo-output $\zeta^0(\tau)$; (7) potential vector $\nu(\tau)$; (8) vector $p(\tau)$; and (9) vector $v^*(\tau) = G(\tau)x^*(\tau)$.

The problem to be solved by the optimal controller at time $\tau+h$ is represented in functional terms as follows:

$$\begin{gathered}\sum_{t\in T_u(\tau)} c'(t)u(t) \to \max,\\ \widetilde{g}_*(\tau+h) \le \sum_{t\in T_u(\tau)} D(t)u(t) \le \widetilde{g}^*(\tau+h),\\ u^*(\tau) \le u(\tau) \le u^*(\tau), \quad u_* \le u(t) \le u^*, \quad t \in T_u(\tau+h);\end{gathered} \tag{22}$$

where $\widetilde{g}_*(\tau+h) = g_*(\tau) - \Delta g(\tau)$, $\widetilde{g}^*(\tau+h) = g^*(\tau) - \Delta g(\tau)$, $\Delta g(\tau) = \int_\tau^{\tau+h} G(t)w^*(t)dt$.

We obtain from the equality

$$x^*(\tau+h) = F(\tau+h)F^{-1}(\tau)x^*(\tau) + \int_\tau^{\tau+h} F(\tau+h)F^{-1}(t)B(t)dt u^*(\tau) + \int_\tau^{\tau+h} F(\tau+h)F^{-1}(t)w^*(t)dt$$

that

$$\int_\tau^{\tau+h} F^{-1}(t)w^*(t)dt = F^{-1}(\tau+h)x^*(\tau+h) - F^{-1}(\tau)x^*(\tau) - \int_\tau^{\tau+h} F^{-1}(t)B(t)dt\ u^*(\tau).$$

Hence, $\Delta g(\tau) = v^*(\tau+h) - v^*(\tau) - D(\tau)u^*(\tau)$.

By integrating system (8) with the initial condition $G(\tau)$ over the interval $[\tau, \tau+h]$ and measuring the current state $x^*(\tau+h)$ at the time $\tau+h$, we calculate the vector $\Delta g(\tau)$.

Let us assume that $K_{\rm s}(\tau+h) = K_{\rm s}^0(\tau)$ and $\zeta_{\rm s}(\tau+h) = \zeta_{\rm s}^0(\tau)$ and determine the support values of the pseudocontrol $\omega_{\rm s}(\tau+h) = (\omega_j(t\,|\,\tau+h), \{j,t\} \in S_{\rm s}(\tau+h))$ from the equation

$$D_{\rm s}(\tau)\omega_{\rm s}(\tau+h) = \zeta_{\rm s}(\tau+h) - v_{\rm s}^*(\tau) - \Delta g_{\rm s}(\tau) - p_{\rm s}(\tau)$$

and the pseudo-output

$$\zeta(\tau+h) = D_{|{\rm s}|}(\tau)\omega_{\rm s}(\tau+h) + v^*(\tau) + \Delta g(\tau) + p(\tau).$$

If the inequality

$$u_{*j} \le \omega_j(t \,|\, \tau + h) \le u_j^*, \{j, t\} \in S_{\rm s}^0(\tau); \quad g_{*i} \le \zeta_i(\tau + h) \le g_i^*, \quad i \in I_{\rm ns}$$

is satisfied and $\{j, \tau\} \notin S_{\rm s}(\tau + h)$, $j \in J$, then we assume that $K_{\rm s}^0(\tau + h) = K_{\rm s}(\tau + h) = K_{\rm s}^0(\tau)$;

$$u_j^*(\tau + h) = \begin{cases} \omega_j(\tau + h|\tau + h), \text{ if } \{j, \tau + h\} \in S_{\rm s}^0(\tau) \\ u_j^*, \text{ if } \gamma_j(\tau) = 1 \wedge \{j, \tau + h\} \notin S_{\rm ns0}(\tau) \text{ or } \gamma_j(\tau) = -1 \wedge \{j, \tau + h\} \in S_{\rm ns0}(\tau) \\ u_{*j}, \text{ if } \gamma_j(\tau) = -1 \wedge \{j, \tau + h\} \notin S_{\rm ns0}(\tau) \text{ or } \gamma_j(\tau) = 1 \wedge \{j, \tau + h\} \in S_{\rm ns0}(\tau). \end{cases}$$

The information stored for the instant τ is transformed for the instant $\tau + h$ as follows. We assume that $D_{|{\rm s}|}(\tau + h) = D_{|{\rm s}|}(\tau)$, $\nu(\tau + h) = \nu(\tau)$, $p(\tau + h) = p(\tau) - D(\tau)u^*(\tau)$, $\zeta^0(\tau + h) = \zeta^0(\tau)$ and, instead of $\psi_c(\tau)$, $G(\tau)$, $D(\tau)$, and $v^*(\tau)$, calculate and store $\psi_c(\tau+h)$, $G(\tau+h)$, $D(\tau+h)$, and $v^*(\tau+h)$. If $\{j, \tau+h\} \in S_{\rm ns0}(\tau)$, then we assume that $S_{\rm ns0}(\tau+h) = S_{\rm ns0}(\tau) \setminus \{j, \tau+h\}$, $k(j \,|\, \tau+h) = k(j \,|\, \tau) - 1$ and re-enumerate the points of the set $T_{\rm s}(j \,|\, \tau)$. For $\{j, \tau + h\} \in S_{\rm s}^0(\tau) \bigcup S_{\rm ns0}(\tau)$, we assume that $\gamma_j(\tau + h) = -\gamma_j(\tau)$.

Otherwise, to construct the optimal support $K_{\rm s}^0(\tau+h)$, we make use of the dual method described in Section 7. If in doing so $\{j, \tau\} \in S_{\rm s}(\tau + h)$ for some $j \in J$, then at the first iterations we remove $\{j, \tau\}$ from the initial support $K_{\rm s}(\tau + h)$, and then all operations in the iterations are carried out on the set $T_u(\tau + h)$, which prevents $\{j, \tau\}$ from entering the optimal support $K_{\rm s}^0(\tau + h)$. If in the course of iterating the dual method there exists no final step $\sigma > 0$ for which the function $\delta(\sigma, t)$, $t \in T_u(\tau + h)$, then problem (22) has no admissible controls, that is, the current state $x^*(\tau + h)$ exceeded the bounds of the admissible set.

Frequent occurrence of the situation $\{j, \tau\} \in S_{\rm s}^0(\tau)$ is indicative of the sliding mode in the closed system (2). The reader is referred to [13] for the methods of regularizing the sliding mode within the framework of the approach under consideration. The idea of feasibility of the above method of optimal design for systems of sufficiently high order can be gained from the example in Section 9.

Note 1. As was assumed above, the initial state x_0 of the system is known and its optimal open-loop control $u^0(t \,|\, t_*, x_0)$, $t \in T_u$, can be constructed in advance. No basic difficulties are encountered also in the case where before starting control it is only known that x_0 belongs to the bounded set $X_0 \subset R^n$.

Note 2. A realization of the continuous optimal feedback based on solving the defining equations by the Newton method was described in [8]. The discrete feedback introduced in [14] was realized there by the methods of linear programming, which prevents one from taking into account all specific features of problem (1). The above realization of the discrete feedback reduces dramatically laboriousness of the support correction, which increases the order of the control systems yielding to the computer-aided optimal design.

9. EXAMPLE

Let us consider the problem

$$\int_0^{15} \sum_{j=1}^{3} u_j(t) \; dt \to \min, \tag{23}$$

$$\dot x_1 = x_3,\ \dot x_2 = x_4,\ \dot x_3 = -x_1 + x_2 + u_1 - u_3,\ \dot x_4 = 0,\ x_1 - 1.02x_2 - u_2 + u_3;$$

$$x_1(0) = 2, \quad x_2(0) = 1, \quad x_3(0) = 7, \quad x_4(0) = 5;$$

$$x_1(15) = 0, \quad x_2(15) = 0, \quad x_3(15) = 0, \quad x_4(15) = 0,$$

$$0 \le u_j(t) \le 1, \quad j = \overline{1, 3}, \quad t \in T = [0, 15[$$

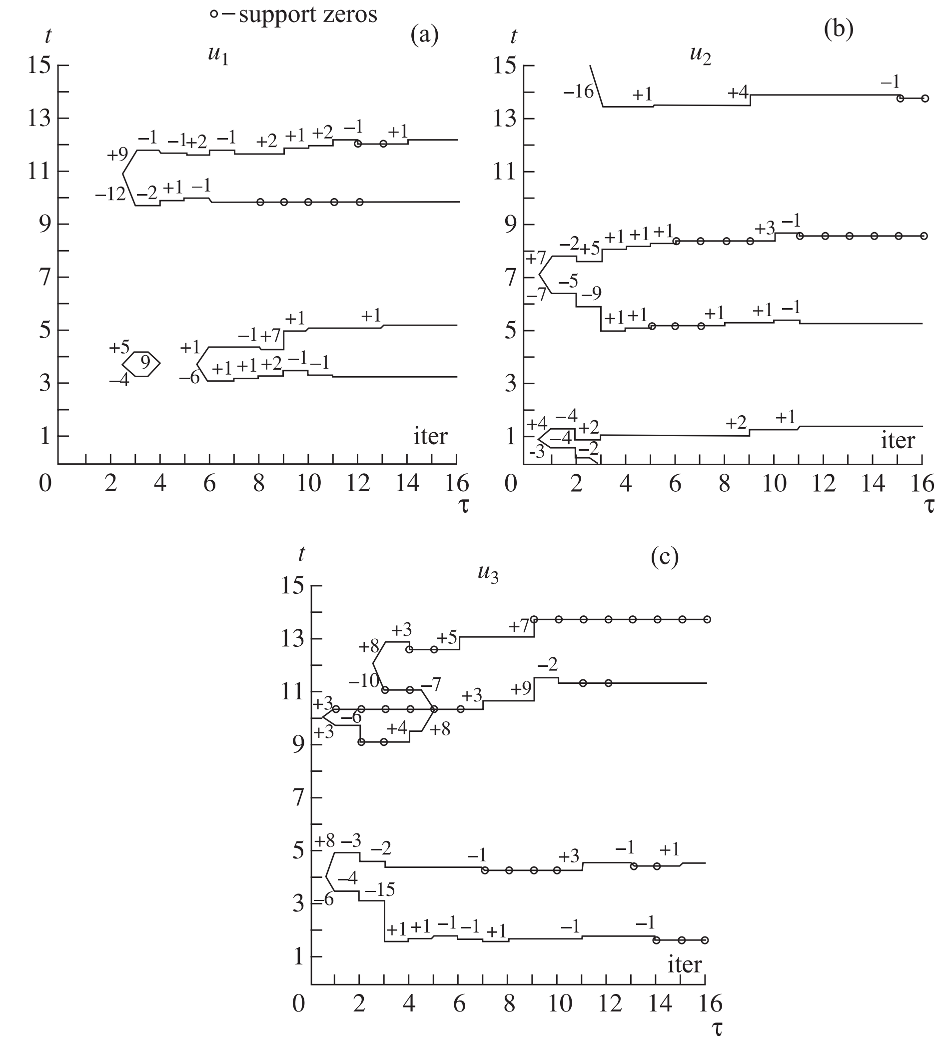

Fig. 1.

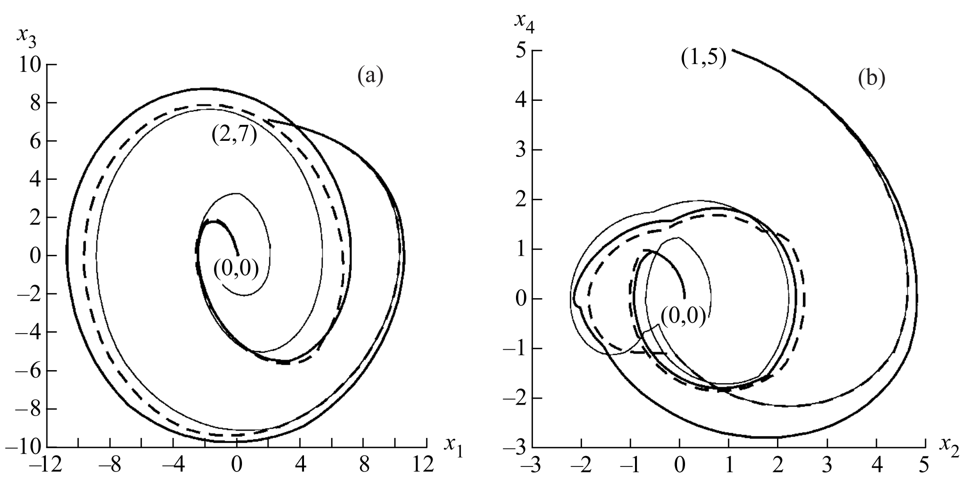

Fig. 2.

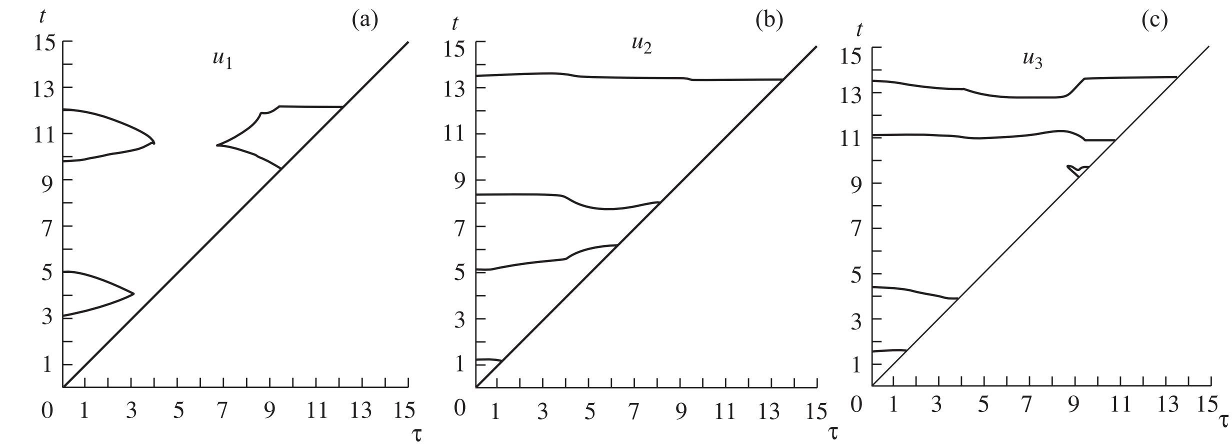

Fig. 3.

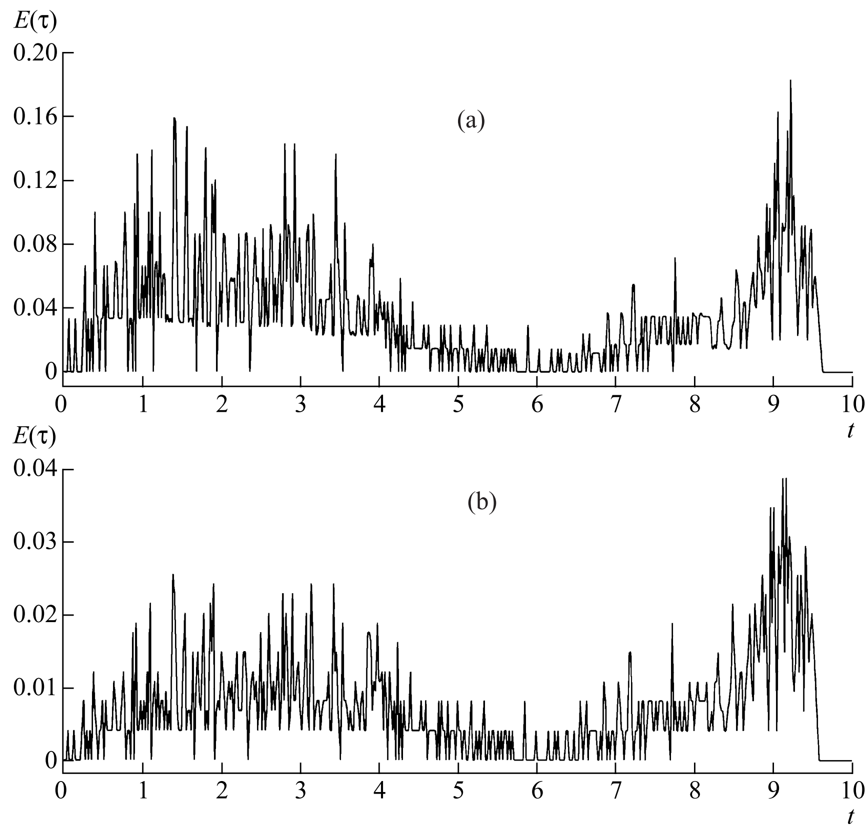

Fig. 4.

illustrating the proposed algorithms. In the class of discrete controls with the time-slotting period $h = 0,1$, it was solved using the dual method of Section 7. The empty support $K_s = \varnothing$ was chosen as the initial one. The problem was solved in sixteen iterations (changes of supports). The optimal value of the performance index was 15.269607. The optimal support had the form $K_s^0 = \{I_s^0, S_s^0\}$: $I_s^0 = \{1, 2, 3, 4\}$, $S_s^0 = \{\{2; 8, 6\}; \{2; 13, 8\}; \{3; 1, 6\}; \{3; 13, 6\}\}$. Figure 1 shows motions of the support and nonsupport zeros of the components of the co-control $\Delta_j(t)$, $j = \overline{1,3}$ in the course of iterations. As can be seen from it, at the beginning of solution of problem (23) the co-control components have no zeros. At the first iteration, the second and third components get four zeros each. After the ninth iteration, the motions of the support and nonsupport zeros are insignificant, which means that at the subsequent iterations the qiasicontrols generate trajectories which with high precision satisfy the terminal constraints on problem (23). Laboriousness of constructing the program solution of problem (23) was 4.053.

Let us construct the positional solution of problem (23). We assume that the dynamic system under consideration is subjected to a sectionally continuous perturbation, and as the result the actual behavior of the system obeys the equations

$$\dot{x}_1 = x_3, \quad \dot{x}_2 = x_4, \quad \dot{x}_3 = -x_1 + x_2 + u_1 - u_3,$$
$$\dot{x}_4 = 0.1x_1 - 1.02x_2 - u_2 + u_3 + w.$$

Let the perturbation realized in the course of control be as follows:

$$w^*(t) = -0.5\,\sin(t/2), \quad t \in [0; 9.5]; \quad w^*(t) \equiv 0, \quad t \in]9.5; 15].$$

It is unknown to the controller, but at each instant $t \in T_u$ it knows the current state of the system $x^*(t)$.

Figure 2 depicts the projections on the phase planes x_1x_3 and x_2x_4 of the trajectories of system (23) generated by (i) the optimal open-loop control $u^0(t)$, $t \in T$, under no perturbation (solid bold line), (ii) the control $u^0(t)$, $t \in T$, with the perturbation $w^*(t)$, $t \in T$, (solid thin line), and (iii) the optimal feedback under the perturbation $w^*(t)$, $t \in T$, (dashed line). Displacements of the elements of the sets $S_{\rm s}(\tau)$ and $S_{\rm ns0}(\tau)$, $\tau \in T_u$, are shown in Fig. 3. Figure 4 shows the laboriousness of correcting the supports by the dual method for $h = 0,02$ with sequential (Fig. 4a) and parallel (Fig. 4b) preparation of the additional information (15). As can be seen from Fig. 4, even the first method requires integration of the adjoint system over an interval not exceeding 20% of the control interval. The existing computers can cope with this job in less than $h = 0.02$ s. Laboriousness of constructing the program solution of problem (23) is of no basic importance because it is constructed before controlling the physical system and the time required to construct it can be regarded as unlimited.

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