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Optimal Control of Nonlinear Systems

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Abstract—The construction of optimal closed loop and open loop solutions to a terminal optimal control problem for nonlinear dynamical systems with a linear objective functional and linear terminal constraints is considered. Locally optimal controls are constructed by correcting the solution to a linearized problem by the small parameter technique. Global optimization is also carried out in two steps. First, a solution to a piecewise linear approximation of the original problem is constructed, and then it is improved using an asymptotic technique. The results are illustrated by solving the problem of damping a simple pendulum and the problem of control of a nonlinear system governed by the Duffing equation.

1. INTRODUCTION

Optimal control problems for nonlinear systems are among the most difficult problems in the mathematical theory of control processes [1, 2]. Their difficulty is largely explained by the great variety of nonlinear systems. In essence, every nonlinear system presents an independent object for investigation and possesses certain specific features. It is not by chance that until the present time only general existence theorems and necessary optimality conditions have been proved, but no efficient constructive optimization methods have been developed. In the literature, major effort is focused on the open loop solution of deterministic optimal control problems, which are, by definition, aimed at revealing the potential capabilities of control systems and are not designed for controlling real-life dynamical systems, which inevitably differ from mathematical models and are subject to unknown perturbations. The theory of open loop optimal control is actually a modern version of the calculus of variations. Its purpose is to optimize functionals with respect to function variables subject to closed constraints rather than to control dynamical systems in the sense that is implied by this term by engineers that design control systems. Generally, control implies that there is a dynamic process evolving over time. However, the popularity of optimal control led to the appearance of publications in which "optimal control" problems do not involve time but rather include length, volume, temperature, and so on as variables. Control considered as a purposeful action must respond to changes in the state of the controlled object that occur in the process of control. Thus, we need closed loop, rather than open loop, controls to actually control dynamical systems. Such controls make it possible to effectively counteract perturbations and compensate for the imprecision of mathematical modeling, which are inevitable in real-life control processes. However, issues of designing closed loop solutions (the problem of synthesis of optimal systems) remain poorly understood in the theory of optimal control. In our opinion, one of the main reasons for this situation is the fact that the classical statement of the optimal synthesis problem, which was suggested in the late 1940s and required that the closed loop optimal controls be designed in a closed form, is too "theoretical" ("analytical"). It does not (and could not in the early 1950s) take into account the capabilities of modern computers, which are necessary for solving somewhat complicated applied problems.

In studies [3–5], an approach to the basic problems of constructive optimal control theory was suggested for linear terminal problems, linear problems with intermediate state constraints, and for piecewise linear problems. The aim of this paper is to extend this approach to a special class on nonlinear optimal control problems. More precisely, we assume that the controlled object itself is nonlinear, and all other elements (the cost function and the constraints) are linear. This enables us to simplify the presentation and reduce the size of the paper. We do not consider all possible cases. Extensions and generalizations will be considered separately.

In Section 2, a statement of the terminal optimal control problem is given that involves a single nonlinear element—a nonlinear controlled object. Feasible controls are supposed to be piecewise constant functions with a given set of possible discontinuities. The choice of this type of feasible controls is dictated by the fact that computers are inevitably used in actual control, and they are discretely operating devices. In terms of discrete controls, open loop and closed loop controls are determined. As an ultimate goal, we state the problem of designing an algorithm for an optimal controller able to produce closed loop optimal controls for any particular control process in real time. Section 3 is devoted to a local solution of the problem when all feasible processes are confined in a small neighborhood of the initial state where the linear approximation of the dynamical system is adequate. The solution to the problem is obtained on the basis of two procedures. The first procedure implies that the linear problem is solved by the techniques described in [3]. We stress that the efficiency of these techniques relies on the use of instants of control switch rather than on the conventional primal and dual variables. The switching instants are related both to the primal and the dual variables via the support. The second procedure makes a correction of the results obtained by the first procedure using an asymptotic method. A technique for correcting open loop controls of the linearized problem is presented in [6–8]. It is based on an asymptotic expansion of the instants of control switch rather than on an asymptotic expansion of the primal and dual variables. The choice of instants of control switch enables us to account for direct constraints on controls, which present severe difficulties for other methods. The implementation of the asymptotic method for the construction of a closed loop solution of the problem is different from that in [7]. It turns out that asymptotic expansions can be constructed rather efficiently if the values of certain finite-dimensional functions of a scalar variable are stored at certain instants of time in the process of solving the linearized problem. This spares an additional integration in the correction procedure, since the required integrals are computed by quadrature formulas. The results are illustrated in Subsection 3.4 by solving an optimal control problem for a dynamical system governed by the Duffing equation. An analysis of the optimization results shows that a rather accurate solution can be obtained by solving the linearized problem when the range of variation of the state of the system is rather small. This range can be considerably enlarged by employing the asymptotic correction. On the other hand, the asymptotic correction allows one to considerably improve the accuracy of the solution when the behavior of the linear system is analyzed in a fixed domain.

In Section 4, we investigate the global optimization of nonlinear systems. Again, the process of solution is composed of two procedures. First, a piecewise linear approximation of the nonlinear system is carried out, and a piecewise linear optimal control problem is solved. Piecewise linear approximation of dynamical systems provides a natural way for the approximate solution of nonlinear problems. This method is a particular case of the spline method, which is widely used in many branches of applied mathematics. A detailed description of a method for the optimization of piecewise linear systems can be found in [5]. Then, the solution to the piecewise linear problem is corrected by an asymptotic technique developed for quasilinear systems. The correction technique for the solution to the piecewise linear problem is a generalization of the correction technique developed for linear problems in [7]. This method, which is justified in Subsections 4.3–4.5, has not yet been published. The results of correction with a certain value of the accuracy parameter are taken for the solution to the initial problem. An algorithm implementing the method described is suggested. The efficiency of the algorithm is illustrated by solving the problem of optimal damping of a simple pendulum. The results reported in Subsection 4.7 show that our approach yields a rather accurate solution to the nonlinear problem even though rough piecewise linear approximations of the nonlinear equation are used.

The efficiency (in the sense of [2, 3]) of the proposed methods for the optimization of nonlinear systems depends on the efficiency of the optimization of linear [3] and piecewise linear [5] systems. This is because no additional integration of the primal and dual systems is carried out on the stage of the asymptotic correction. However the requirements for the main computer memory are now higher, since additional information on the behavior of the systems to be optimized is needed for the calculations by quadrature formulas.

In conclusion, we note that we construct open loop and closed loop solutions of the optimization problem for nonlinear systems on the basis of two classical methods—an analog of the dual simplex method in linear programming and the small parameter method. It turned out that an appropriate implementation of these methods makes it possible to obtain satisfactory solutions to basic optimal control problems. In other words, constructive methods in optimal control could have been developed as early as in the beginning of the 1950s, when those problems were first stated, but the fundamental facts of the mathematical theory of optimal processes—the maximum principle [1] and the dynamical programming [2]—had not yet been discovered.

2. STATEMENT OF THE PROBLEM

Let X be a bounded set in an n-dimensional space, $T = [0, t^*]$ be the control time interval, $h = t^*/N$ be the quantization interval, N be a positive integer, and $T_h = \{0, h, \ldots, t^* - h\}$. The function $u(t)$ $(t \in T)$ is called a discrete control (with the quantization interval h) if $u(t) = u(kh)$ for $t \in [kh, (k + 1)h[$ $(k = \overline{0, N - 1})$.

On the set X, consider the problem

$$c'x(t^*) \longrightarrow \max, \quad \dot{x} = f(x) + bu, \quad x(0) = x_0,$$
$$x(t^*) \in X^* = \{x \in \mathbb{R}^n : Hx = g\}, \quad |u(t)| \le 1, \quad t \in T \tag{2.1}$$

in the class of discrete controls. Here, $c, b \in \mathbb{R}^n$, $g \in \mathbb{R}^m$, $H \in \mathbb{R}^{m \times n}$, $\operatorname{rank} H = m < n$, $x = x(t)$ is the state vector of the control system at an instant t, $u = u(t)$ is the value of a scalar control, and $f(x)$ $(x \in X)$ is an infinitely differentiable n-dimensional vector function on $\operatorname{int} X$.

As usual, the control $u(t)$ $(t \in T)$ is called a feasible (open loop) control if $|u(t)| \le 1$ for $t \in T$ and the corresponding trajectory of system (2.1) satisfies the terminal constraint $x(t^*) \in X^*$. A feasible control $u^0(t)$ is called the optimal open loop control for problem (2.1) if the corresponding trajectory $x^0(t)$ $(t \in T)$ maximizes the objective functional; i.e., $c'x^0(t^*) = \max c'x(t^*)$. Here, the maximum is taken over all feasible controls, and $x^0(t)$ $(t \in T)$ is the optimal trajectory.

Before we introduce the concept of closed loop optimal control for problem (2.1), assume that the states of the control system are known not only at the initial time $t = 0$, but will be known at every current time $\tau \in T_h$ in the process of control. Under these assumptions, we embed problem (2.1) in the family of problems

$$c'x(t^*) \longrightarrow \max, \quad \dot{x} = f(x) + bu, \quad x(\tau) = z,$$
$$x(t^*) \in X^* = \{x \in \mathbb{R}^n : Hx = g\}, \quad |u(t)| \le 1, \quad t \in T^\tau = [\tau, t^*], \tag{2.2}$$

which depends on a scalar $\tau \in T_h$ and an n-dimensional vector z.

Let $u^0(t \,|\, \tau, z)$ $(t \in T^\tau)$ be the optimal open loop control for problem (2.2) for (τ, z) and X_τ be the set of states $z \in X$ for which problem (2.2) has an open loop solution with a fixed τ.

The function

$$u^0(\tau, z) = u^0(\tau|\tau, z), \quad z \in X_\tau, \quad \tau \in T_h, \tag{2.3}$$

is called the closed loop (discrete) optimal control.

According to the definitions, the open loop, $u^0(t)$ $(t \in T)$, and closed loop, $u^0(t, x)$ $(x \in X_t, t \in T_h)$, controls are determined on the basis of the a priori information about problem (2.1). Hence, theoretically, they can be constructed before the actual control process starts. As a rule, these solutions cannot be found analytically (in an explicit closed form). A numerical construction of open loop and closed loop solutions presupposes that they should be represented in a tabular form. For an open loop solution to system (2.1), this is an easy problem even for problems of a high dimension. However, tabulating the closed loop solution with a small error leads to the “curse of dimensionality” [9], which can hardly be worked around.

Approximate methods for constructing open loop solutions are described in [2]. A classical method that can be used for finding closed loop solutions is the dynamic programming [9]. In the above statement of the problem, the justification of dynamic programming is easy, since the Bellman equation for the discrete control becomes recurrent:

$$B(\tau, z) = \max_{|u| \le 1} B(\tau + h, x(\tau + h|\tau, z, u)),$$

$$B(t^*, z) = \begin{cases} c'z, & z \in X^*, \\ -\infty, & z \notin X^* \end{cases}$$

where $(x(\tau + h \,|\, \tau, z, u)$ is the state of system (2.1) at the time $\tau + h$ that it takes under the influence of the control $u(t) = u$, $t \in [\tau, \tau + h[$).

The optimal feedback $u^0(\tau, z)$ $(z \in X_\tau, \tau \in T_h)$ is found from the equation

$$B(\tau + h, x(\tau + h, \tau, z, u^0(\tau, z))) = \max_{|u| \le 1} B(\tau + h, x(\tau + h \,|\, \tau, z, u)), \quad X_\tau = \{z \in X : B(\tau, z) \ne -\infty\}.$$

An analysis (see [10]) shows that the “curse of dimensionality” in the synthesis of optimal systems by dynamic programming is caused by the neglect of the fact that transient processes in actual dynamical systems have a bounded rate. Due to a bounded rate, the state of the system changes insignificantly between the times of decision making (when control values are chosen); therefore, there is a time interval when useful calculations can be carried out to correct the control. The amount of calculations that can be done depends on the performance of the available hardware. In dynamic programming, it is implicitly assumed that transient processes can have an infinite rate. For this reason, all calculations are performed for all possible states of the systems before the actual control process begins, and the results of those calculations are used for control without any additional corrections.

Taking into account the difficulties described above, a workaround for the curse of dimensionality was suggested in [11]. The idea is as follows. In the process of solving an applied problem, the feedback (2.3) is calculated on the basis of the mathematical model (2.1). Then, this feedback is used to control the real system rather than its mathematical model. Real systems differ from "ideal" models (2.1) in the imprecision of mathematical modeling and are subject to perturbations, which cannot be taken into account in advance (before the control process starts). Assume that the behavior of a real system closed by feedback (2.3) is described by the equation

$$\dot{x} = f(x) + bu^0(t, x) + w, \quad x(0) = x_0, \tag{2.4}$$

where $w = w(t, x)$ ($x \in X$, $t \in T$) is an unknown perturbation that is realized as a piecewise continuous n-dimensional vector function $w(t) = w(t, x(t))$ ($t \in T$) along every continuous function $x = x(t)$ ($t \in T$).

In the classical definition of a closed loop solution (for piecewise continuous measurable controls) the following mathematical problem arises: how the solution of the differential equation (2.4) with a discontinuous right-hand side should be interpreted. Due to this discontinuity, the classical solution to Eq. (2.4) can not exist; the use of generalized solutions in Filippov's sense does not eliminate this difficulty.

In this paper, we interpret the trajectory of system (2.4) as the trajectory of the equation

$$\dot{x} = f(x) + bu^*(t) + w(t), \quad x(0) = x_0, \tag{2.5}$$

under the control

$$u^*(t) = u^0(kh, x(kh)), \quad t \in [kh, (k+1)h[, \quad k = \overline{0, N-1}. \tag{2.6}$$

Obviously, no problem of the existence of the classical solution arises in this case.

It is seen from Eq. (2.5) that, in each concrete control process corresponding to a particular initial state x_0 and a particular perturbation $w(t)$ ($t \in T$), only the realization (2.6) of feedback (2.2) along the continuous trajectory $x(t)$ ($t \in T$) is used.

Assume that the time required for the calculation of the value $u^*(\tau)$ at every instant $\tau \in T_h$ does not exceed h. Then, we say that the feedback is realized in real time. A device that can perform this calculation is called an optimal controller for problem (2.1). Thus, the synthesis of the optimal closed loop control is reduced to designing an algorithm for the optimal controller.

In what follows, we will make a distinction between the local and global optimization of nonlinear systems.

3. LOCAL OPTIMIZATION OF NONLINEAR SYSTEMS

Let x^* be a point in X, $\hat{f}(x) = A(x - x^*) + d$, $d = f(x^*)$, and $A = \partial f(x)/\partial x|_{x = x^*}$. The number

$$\delta = \max_{x \in X} \| f(x) - \hat{f}(x) \| / \| f(x) \| \tag{3.1}$$

is called the error of the linear approximation of $f(x)$ on the set X.

If the function $\hat{f}(x) = Ax + a$ ($x \in X$ ($a = -Ax^* + d$)) is a good approximation to $f(x)$ ($x \in X$), i.e., if δ is sufficiently small, then the equation

$$\dot{x} = f(x) + bu \tag{3.2}$$

is equivalent to the nearly linear equation

$$\dot{x} = \hat{f}(x) + \delta g(x) + bu, \tag{3.3}$$

where $g(x) = (f(x) - Ax - a)/\delta$ ($x \in X$). (Certainly, every nonlinear system (3.2) on any set X can be represented in the form (3.3) with an arbitrary small δ. However, the functions $Ax + a$ and $g(x)$ are not always of the same order of magnitude on X.)

In this section, we solve problem (2.1) on a set X for which δ is small.

The solution of problem (2.1) is divided into two steps: (1) solution of a linearized problem and (2) correction of the solution to the linearized problem.

3.1. Solution of the Linearized Problem

The linearized problem has the form

$$c'x(t^*) \longrightarrow \max, \quad \dot{x} = Ax + a + bu, \quad x(0) = x_0, \quad x(t^*) \in X^*, \quad |u(t)| \le 1, \quad t \in T. \tag{3.4}$$

An algorithm for the construction of the open loop and closed loop solutions to problem (3.4) can be found in [3]. Here, we give some definitions and results from [3] that are necessary for the further consideration.

The basic tool of the method described in [3] is the support (or working basis). Recall that the set $T_{\sup} = \{t_l, l = \overline{1, m}\} \subset T_h$ is called a support of problem (3.4) if the support matrix $D_{\sup} = (d(t), t \in T_{\sup})$ is nonsingular. Here,

$$d(t) = \int_t^{t+h} G(\vartheta)bd\vartheta, \quad \dot{G} = -GA, \quad G(t^*) = H.$$

The following elements are associated with the support:

1. A vector of potentials ν that is a solution to the equation

$$D'_{\sup}\nu = c_{\sup}, \text{ where } c_{\sup} = (c(t), t \in T_{\sup}), \quad c(t) = \int_t^{t+h} \psi'_c(\vartheta)bd\vartheta, \quad \dot{\psi}_c = -A'\psi_c, \quad \psi_c(t^*) = c.$$

2. A cocontrol is defined as

$$\Delta(t) = \int_t^{t+h} [\psi'_c(\vartheta) - \nu'G(\vartheta)]bd\vartheta, \quad t \in T_h.$$

3. A pseudocontrol $\omega(t)$ ($t \in T_h$) is defined so that its nonsupport values $\omega(t)$, $t \in T_{\text{nsup}} = T_h \backslash T_{\sup}$ satisfy the relations

$$\omega(t) = \operatorname{sgn}\Delta(t), \quad \text{if} \quad \Delta(t) \ne 0, \quad \omega(t) \in [-1, 1], \quad \text{if} \quad \Delta(t) = 0, \quad t \in T_{\text{nsup}}. \tag{3.5}$$

The support values $\omega(t)$ ($t \in T_{\sup}$) of the pseudocontrol are determined from the equation

$$\sum_{t \in T_{\sup}} d(t)\omega(t) + \sum_{t \in T_{\text{nsup}}} d(t)\omega(t) = \hat{g}, \quad \hat{g} = g - G(0)x_0 - \int_0^{t^*} G(\vartheta)ad\vartheta.$$

Although the pseudocontrol is defined on the discrete set T_h, we assume that it (and the cocontrol below) is extended to all other points of the interval T in the natural way as $\omega(s) \equiv \omega(t)$ for $s \in [t, t+h[$, $t \in T_h$.).

4. A quasi-control is defined as

$$\tilde{\omega}(t) = \begin{cases} \omega(t), & |\omega(t)| \le 1, \\ \operatorname{sgn}\omega(t), & |\omega(t)| > 1, \quad t \in T_h. \end{cases}$$

5. The norm of the residual of intermediate constraints on the quasi-control is defined as

$$\tilde{g}(T_{\sup}) = \|g - H\tilde{x}(t^*)\|,$$

where $\tilde{x}(t)$ ($t \in T$) is the trajectory of system (3.4) corresponding to the quasi-control $\tilde{\omega}(t)$ ($t \in T$).

For given ε_1, $\varepsilon_2 \ge 0$, the available control $u(t)$ ($t \in T$) is called an $\varepsilon_1\varepsilon_2$-solution to problem (3.4) if the corresponding trajectory $x(t)$ ($t \in T$) satisfies the inequality

$$c'x^0(t^*) - c'x(t^*) \le \varepsilon_1,$$

where $x^0(t)$ ($t \in T$), and the norm of the residual of the terminal constraints $\tilde{g}(u(\cdot)) = \|g - Hx(t^*)\|$ satisfies the inequality $\tilde{g}(u(\cdot)) \le \varepsilon_2$ for the control $u(t)$ ($t \in T$). If the pseudocontrol $\omega(t)$ ($t \in T_h$) constructed on the basis of $T_{\sup}$ satisfies the inequality $|\omega(t)| \le 1$ ($t \in T_{\sup}$), then $\omega(t)$ ($t \in T_h$) is an optimal control for problem

(3.4). If, for a given $\varepsilon_2 \geq 0$, the inequality $\|\tilde{g}(T_{sup})\| \leq \varepsilon_2$ is satisfied for the pseudocontrol $\tilde{\omega}(t)$ $(t \in T_h)$ constructed on the basis of T_{sup}, then $\tilde{\omega}(t)$ $(t \in T_h)$ is a $0\varepsilon_2$-solution to problem (3.4).

In [3], dynamic implementations of primal and dual adaptive methods for problem (3.4) were described. Here, we briefly review the essence of the dual method.

In the iteration process of the dual adaptive method, the support T_{sup} is replaced by a new support $\bar{T}_{sup}$ so that $\tilde{g}(\bar{T}_{sup}) \leq \tilde{g}(T_{sup})$. The instant $t^0 \in T_{sup}$ such that $|\omega(t^0)| = \max|\omega(t)|$ $(t \in T_{sup})$ is removed from the support.

In order to determine an instant to be added to the support, the following operations are carried out.

1. The variation of the vector of potentials, $\Delta\nu$, is calculated.

2. For the dual problem of (3.4), the rate of change of the objective functional is calculated.

3. A short step is calculated along the direction $\Delta\nu$ that ensures the appearance of a new zero of the cocontrol being varied. This step causes a positive jump of rate in the change of the dual objective functional.

4. Auxiliary information stored in the computer memory is transformed.

5. Operations 2–4 are repeated until the rate of change of the dual objective functional becomes a nonnegative value.

6. The last zero $\tilde{t}^{K_0}$ of the varied cocontrol is added to the modified support $\bar{T}_{sup}$ instead of the support element t_0.

While modifying the support, additional information is used. It includes the set of nonsupport zeros of the cocontrol $T_{nsup0} = \{t \in T_{nsup} \backslash 0 : \Delta(t-h)\Delta(t) < 0\}$, the set $T_{sup.nsup} = T_{sup} \cup T_{nsup0} \cup \{0, t^*\} = \{t_k, k \in K \cup k^*+1\}$ $(K = \{0, 1, \ldots, k^*\})$, the number

$$\gamma = \begin{cases} \operatorname{sgn}\Delta(0), & 0 \notin T_{sup}, \\ \operatorname{sgn}\Delta(h), & 0 \in T_{sup}; \end{cases}$$

and the vector

$$p = \sum_{t \in T_{nsup}} \int_t^{t+h} G(\vartheta) b \, d\vartheta \, \omega(t) + \int_0^{t^*} G(\vartheta) d\vartheta \, a.$$

According to [3], the following data are stored and transformed in the course of iteration: (1) the support T_{sup}; (2) the set T_{nsup0}; (3) the support matrix D_{sup}; (4) the quantities $G(t)$, $\psi_c(t)$ $(t \in T_{sup.nsup} \backslash t^*)$; (5) the vector p; (6) the number γ and the support values of the pseudocontrol $\omega(t)$ $(t \in T_{sup})$; and (7) the vector of potentials ν.

Thus, the algorithm described in [3] produces the optimal support T^0_{sup} of problem (3.4), the corresponding optimal control $u^0(t)$ $(t \in T)$, and the data (1)–(7), which will be used for the asymptotic correction of the solution to problem (3.4).

At small δ, the optimal control constructed for the linear system (3.4) and used to control the nonlinear system (2.1) works rather satisfactorily (see the example below).

However, in order to extend the set X for which a solution to problem (2.1) can be constructed with a given accuracy, we invoke the following procedure.

3.2. Asymptotic Correction of the Solution to the Linear Problem

We embed problem (2.1) in the family of problems depending on a small parameter $\mu \longrightarrow 0$ (a quasilinear optimal control problem)

$$\begin{gathered} c'x(t^*) \longrightarrow \max, \quad \dot{x} = Ax + a + \mu g(x) + bu, \quad x(0) = x_0, \\ x(t^*) \in X^*, \quad |u(t)| \leq 1, \quad t \in T. \end{gathered} \tag{3.6}$$

(In the quasilinear optimal control problem, μ plays the role of a variable rather than a parameter; i.e., it is akin to the variable t rather than to the parameters A and b.)

For $\mu = \delta$, problem (3.6) includes problem (2.1). An asymptotic method for the construction of an open loop solution to problem (3.6) was described in [6–8]. Here, we give some definitions and results that are required for the further consideration.

For a fixed parameter μ, the piecewise continuous function $u(t, \mu)$ $(t \in T)$ satisfying the inequality $|u(t, \mu)| \le 1$ is called an available control. An available control is called a feasible open loop control if the corresponding trajectory $x(t, \mu)$ $(t \in T)$ of system (3.6) satisfies the constraint $x(t^*, \mu) \in X^*$. A feasible control $u^0(t, \mu)$ $(t \in T)$ that minimizes the objective functional of problem (3.6) is called an optimal open loop control ($u^0(t, \mu)$, $(t \in T)$ is a solution to the parametric optimal problem in which, in contrast to the quasilinear optimal problem, μ is a parameter (as A and b), and it is the dependence of the classical (rather than asymptotic) solutions on this parameter that is analyzed.) For a given positive integer s, the family $u^s(t) = \{u^s_\mu(t), \mu \longrightarrow 0\}$ $(t \in T)$ of available controls is called an symptotically s-optimal open loop control for the quasilinear problem (3.6) if the following asymptotic equalities hold for the trajectories $x^s_\mu(t)$ $(t \in T, \mu \longrightarrow 0)$ corresponding to the controls $u^s_\mu(t)$ $(t \in T, \mu \longrightarrow 0)$: $c'x^0(t^*, \mu) - c'x^s_\mu(t^*) = o(\mu^s)$, $Hx^s_\mu(t^*) - g = o(\mu^s)$ as $\mu \longrightarrow 0$.

The key elements of the optimal control $u^0(t, \mu)$ $(t \in T)$ for problem (3.6) are the switching points $t_j(\mu)$ $(j = \overline{1, k^*})$ and the vector of potentials $\nu(\mu)$. Following [7], in order to construct an asymptotically s-optimal control, we calculate the coefficients $z_k = (t^k_j, j = \overline{1, k^*};\ \nu^k_i, i = \overline{1, m})$ $(k = \overline{0, s})$ of s-degree Taylor's polynomials of the functions $t_j(\mu)$, $\nu(\mu)$:

$$t^s_j(\mu) = \sum_{k=0}^{s} \mu^k t^k_j, \quad j = \overline{1, k^*}, \quad \nu^s_i(\mu) = \sum_{k=0}^{s} \mu^k \nu^k_i, \quad i = \overline{1, m}.$$

The coefficients $z_0 = (t^0_j, j = \overline{1, k^*}, \nu^0_i, i = \overline{1, m})$ are found by solving the basic problem (3.4) in the class of piecewise continuous functions. They are constructed by solving the refinement equations (see [3])

$$R_0(z) = 0 \tag{3.7}$$

in the variables $z = (t_j, j = \overline{1, k^*}, \nu_i, i = \overline{1, m})$, where

$$R_0(z) = \begin{bmatrix} \sum_{j=0}^{k^*} (-1)^j \gamma \int_{t_j}^{t_{j+1}} G(t) b\,dt - \hat{g}, \\ (\psi'_c(t_j) - \nu' G(t_j)) b,\ j = \overline{1, k^*} \end{bmatrix}. \tag{3.8}$$

System (3.7) is solved by Newton's method. The approximation z^l_0 of the elements z_0 is calculated by the formula

$$z^l_0 = z^{l-1}_0 - I^{-1}_0(z^{l-1}_0) R_0(z^{l-1}_0), \quad l = \overline{1, l_0},$$

where $I_0(z)$ is the Jacobian of system (3.7):

$$I_0(z) = \begin{pmatrix} 2(-1)^{j-1} \gamma G(t_j) b,\ j = \overline{1, k^*} & 0_{m \times m} \\ \operatorname{diag}((\nu' G(t_j) - \psi'_c(t_j)) Ab,\ j = \overline{1, k^*}) & -(G(t_j) b)',\ j = \overline{1, k^*} \end{pmatrix}.$$

Then, we define $z_0 = z^{l_0}_0$, where $z^{l_0}_0$ is the approximation that ensures the inequality $\|R_0(z^{l_0}_0)\| \le \varepsilon$ for the given $\varepsilon \ge 0$.

The initial approximation $z^0_0 = (t^{0(0)}_j, j = \overline{1, k^*}, \nu^{0(0)}_i, i = \overline{1, m})$ is composed of the elements of the set $T^0_{\text{nsup}} \backslash \{0, t^*\}$ $(t^0_0 = 0, t^0_{k^*+1} = t^*)$ and of the optimal vector of potentials obtained by solving problem (3.4) in the class of discrete controls. The quantities $G(t^{0(0)}_j)$, $\psi_c(t^{0(0)}_j)$ $(j = \overline{1, k^*})$, and the vector p are used to calculate $R_0(z_0^0)$ and $I_0(z_0^0)$. At every iteration step of Newton's method, the functions $G(t)$ and $\psi_c(t)$ are integrated over the intervals $[t_j^{0(l-1)}, t_j^{0(l)}]$ ($j = \overline{1, k^*}$). If the integration is performed in parallel, the total cost of the refinement procedure is

$$A_{\mathrm{ref}} = \sum_{l=1}^{l_0} \max_{j = \overline{1, k^*}} \left| t_j^{0(l)} - t_j^{0(l-1)} \right| / t^*.$$

According to [7], the coefficients z_k ($k = \overline{1, s}$) are calculated one after another by solving the systems of linear equations

$$I_0(z_0)z_1 = -R_1(z_0),$$

$$I_0(z_0)z_2 = -\frac{\partial R_1}{\partial z}(z_0)z_1 - \frac{1}{2}z_1'\frac{\partial^2 R_0}{\partial z^2}(z_0)z_1 - R_2(z_0),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

where

$$R_1(z) = \begin{bmatrix} Hx_1(t^*|t_1, \ldots, t_{k^*}), \\ \psi_1'(t_j|z)b, \ j = \overline{1, k^*} \end{bmatrix},$$

$$R_2(z) = \begin{bmatrix} Hx_2(t^*|t_1, \ldots, t_{k^*}), \\ \psi_2'(t_j|z)b, \ j = \overline{1, k^*} \end{bmatrix},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and the functions $x_k(t) = x_k(t \mid t_1, \ldots, t_{k^*})$, $\psi_k(t) = \psi_k(t \mid z)$ ($k = \overline{1, s}$) satisfy the equations

$$\dot{x}_1 = Ax_1 + g(x_0(t)), \quad x_1(0) = 0,$$

$$\dot{x}_2 = Ax_2 + \frac{\partial g(x_0(t))}{\partial x}x_1(t), \quad x_2(0) = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\dot{\psi}_1 = -A'\psi_1 - \frac{\partial g}{\partial x}(x_0(t))\psi_0(t), \quad \psi_1(t^*) = 0,$$

$$\dot{\psi}_2 = -A'\psi_2 - \frac{\partial H(x_0(t), \psi_1(t))}{\partial x} - \frac{\partial^2 H(x_0(t), \psi_0(t))}{\partial x^2}x_1(t), \quad \psi_2(t^*) = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

where $H(x, \psi) = \psi' g(x)$, $x_0(t)$, $\psi_0(t)$ ($t \in T$) are the optimal trajectory and cotrajectory of the basic problem.

Having calculated the coefficients t_j^k ($j = \overline{1, k^*}$), ν_i^k ($i = \overline{1, m}$, $k = \overline{1, s}$) and setting $\mu = \delta$, we obtain

$$t_j^s(\delta) = \sum_{k=0}^{s} \delta^k t_j^k, \quad j = \overline{1, k^*}, \quad \nu_i^s(\delta) = \sum_{k=0}^{s} \delta^k \nu_i^k, \quad i = \overline{1, m}.$$

The control $u^s(t) = (-1)^j \gamma$ ($t \in [t_{j-1}^s(\delta), t_j^s(\delta)[$, $j = \overline{1, k^* + 1}$) is taken for the open loop control of problem (2.1).

As a rule, there are no severe restrictions on the execution time for the construction of the open loop control. For this reason, we do not discuss here the calculation of the required values of $x_k(t)$, $\psi_k(t)$ ($t \in T$, $k =$

$\overline{1, s}$), and their derivatives. This issue will be discussed in the next subsection in which the construction of a closed loop control is described.

3.3. Realization of an Optimal Feedback

It was mentioned above that, in a particular control process, the behavior of a nonlinear system is described by Eq. (2.5). The realization of the optimal feedback $u^*(\tau)$ is to be calculated at every current instant $\tau \in T_h$. As in the construction of an open loop solution, one can use both the solution to the linearized problem or its asymptotic correction.

The construction of an optimal feedback for the linearized problem is described in [3]. It is based on the dual method. At every particular instant $\tau \in T_h$, the problem

$$c'x(t^*) \longrightarrow \max, \quad \dot{x} = Ax + a + bu, \quad x(\tau) = x^*(\tau), \quad x(t^*) \in X^*, \quad |u(t)| \le 1, \quad t \in T^\tau \tag{3.9}$$

is considered in which the initial state $x(\tau)$ is set to the state $x^*(\tau)$ of the nonlinear system (2.5) reached under the effect of the control $u^*(t)$ ($t \in [0, \tau[$) produced by the controller and the actual (realized) perturbation $w^*(t)$ ($t \in [0, \tau[$).

Let $u^0(t \mid \tau, x^*(\tau))$ ($t \in T^\tau$) be the optimal open loop control for problem (3.9) at the state $(\tau, x^*(\tau))$. Then, the control $u^*(\tau)$ takes the form $u^*(\tau) = u^0(\tau|\tau, x^*(\tau))$. The open loop control $u^0(t \mid \tau, x^*(\tau))$ ($t \in T^\tau$) is constructed as the quasi-control corresponding to the optimal support $T^0_{\sup}(\tau)$ which, in turn, is constructed by the dual method as a result of a transformation of data (1)–(7) stored at the instant $\tau - h$.

Now, we describe a technique for the construction of $u^*(\tau)$ ($\tau \in T$) based on asymptotic expansions.

To define the concepts pertaining to closed loop optimal control, we consider the family of problems (3.6) as an element of a more general family,

$$\begin{gathered} c'x(t^*) \longrightarrow \max, \quad \dot{x} = Ax + a + \mu g(x) + bu, \quad x(\tau) = z, \\ x(t^*) \in X^*, \quad |u(t)| \le 1, \quad t \in T^\tau, \end{gathered} \tag{3.10}$$

which also depends on $\tau \in T$ and $z \in X$.

Let $u^s(t \mid \tau, z) = \{ u^s_\mu(t \mid \tau, z), \mu \longrightarrow 0\}$ ($t \in T^\tau$) be an s-asymptotically optimal open loop control for the state (τ, z), Ω_s be the set of states (τ, z) for which closed loop asymptotically s-optimal controls $u^s_\mu(\tau, z) = u^s_\mu(\tau \mid \tau, z)$ ($(\tau, z) \in \Omega_s$) exist.

The family $u^s(\tau, z) = \{ u^s_\mu(\tau, z), \mu \longrightarrow 0\}$ ($(\tau, z) \in \Omega_s$) is called the s-optimal closed loop control. Consider the behavior of the system driven by unknown perturbations and the closed loop s-optimal feedback

$$\dot{x} = f(x) + bu^s_\delta(t, x) + w(t), \quad x(0) = x_0. \tag{3.11}$$

Denote by $x^*(t)$ ($t \in T$) the trajectory of Eq. (3.11) corresponding to a realized perturbation $w^*(t)$ ($t \in T$).

The function $u^{s*}(t) = u^s_\delta(t, x^*(t))$ ($t \in T_h$) will be referred to as a realization of the s-optimal feedback, and a device that calculates the values of this function in real time will be called the optimal controller [12]. We take the s-optimal control with a chosen s for an optimal controller solving the local optimization problem for the nonlinear system.

We now describe an algorithm for a 1-optimal controller.

Prior to turning the controller on, we fix N points s_l ($l = \overline{1, N}$) on the interval T and calculate and store $2N$ matrices $F(s_l)$, $F(t^*)F^{-1}(s_l)$ ($l = \overline{1, N}$), where $F(t)$ is an $n \times n$ matrix solution to the initial problem $\dot{F} = AF$, $F(0) = E$.

Assume that the controller have been constructed, has operated during the time interval $[0, \tau]$, and system (3.11) reached the state $x^*(\tau)$ at $\tau \in T_h$. For this state, we solve the basic problem (3.9). The switching points and the vector of potentials $z_0(\tau) = \{ t^0_j(\tau), j = \overline{1, k^*(\tau)}; \nu^0(\tau)\}$ is found by solving Eqs. (3.7), (3.8), where

$z = z(\tau)$, $k^* = k^*(\tau)$, $\gamma = \gamma(\tau)$, and

$$\hat{g} = \hat{g}(\tau) = g - HF(t^*)F^{-1}(\tau)x^*(\tau) - \int_{\tau}^{t^*} HF(t^*)F^{-1}(\vartheta)ad\vartheta.$$

The collection $S(\tau) = \{\gamma(\tau), k^*(\tau)\}$ is called the optimal control structure for the basic problem. If $S(\tau)$ coincides with $S(\tau - h)$, we solve Eq. (3.7) by Newton's method taking the elements $z_0(\tau - h)$, which were constructed at $\tau - h$, as the initial approximation $z_0^0(\tau)$. The rules of transition between adjacent parts with an invariable structure are similar to those described in [11].

In the process of solving the basic problem, we collect and store information required for the calculation of the coefficients of the first approximation

$$\int_{t_j^0(\tau)}^{t_{j+1}^0(\tau)} F^{-1}(\vartheta)bd\vartheta, \quad j = \overline{0, k^*(\tau)}, \qquad \int_{t_{j(l)}^0(\tau)}^{s_l} F(s_l)F^{-1}(\vartheta)bd\vartheta,$$

where $j(l)$ is a number such that $t_{j(l)}^0(\tau) < s_l \le t_{j(l)+1}^0(\tau)$ $(l = \overline{1, N(\tau)})$, $F^{-1}(t_j^0(\tau))$ $(j = \overline{0, k^*(\tau)})$, and

$$\int_{\tau}^{s_l} F(s_l)F^{-1}(\vartheta)ad\vartheta, \; l = \overline{1, N(\tau)}.$$

If the integration is performed in parallel, the cost of calculating these data is

$$A_\tau = \left(h + \sum_{l=1}^{l_0(\tau)} \max_{j = \overline{1, k^*(\tau)}} \left|t_j^{0(l)}(\tau) - t_j^{0(l-1)}(\tau)\right|\right)(t^*)^{-1},$$

where $l_0(\tau)$ is the number of iteration steps in Newton's method.

To calculate $z_1(\tau) = \{t_j^1(\tau), j = \overline{1, k^*(\tau)}; \nu^1(\tau)\}$, we solve the system

$$I_0(z_0(\tau))z_1(\tau) = -R_1(z_0(\tau)), \tag{3.12}$$

where $I_0(z_0(\tau))$ is the Jacobian matrix for Eq. (3.7) constructed in the course of solving the basic problem, and the quantities $x_1(t^*)$ and $\psi_1(t_j^0(\tau))$ $(j = \overline{1, k^*(\tau)})$ required for the formation of the right-hand side of Eq. (3.12) are calculated by solving the equations

$$\begin{aligned} \dot{x}_1 &= Ax_1 + g(x_0(t|\tau)), \quad x_1(\tau) = 0, \\ \dot{\psi}_1 &= -A'\psi_1 - \frac{\partial g}{\partial x}(x_0(t|\tau))\psi_0(t|\tau), \quad \psi_1(t^*) = 0. \end{aligned} \tag{3.13}$$

The idea of a quick calculation of $x_1(t^*)$ and $\psi_1(t_j^0(\tau))$ $(j = \overline{1, k^*(\tau)})$ is based on quadrature formulas

$$x_1(t^*) = \sum_{l=1}^{N(\tau)} F(t^*)F^{-1}(s_l)g(x^0(s_l|\tau))h_l,$$

$$\psi'_1(t_j^0(\tau)) = -\sum_{l = s_l t_j^0(\tau)} \psi'_0(s_l|\tau)\frac{\partial g'}{\partial x}(x^0(s_l|\tau))F(s_l)F^{-1}(t_j^0(\tau))h_l.$$

Here, $N(\tau)$ is the number of points s_l $(l = \overline{1, N})$ that were fixed in advance and belong to the interval T^τ, and h_l $(l = \overline{1, N})$ are the coefficients that depend on the quadrature formula (in the simplest case, $h_l = \text{const}$ for $l = \overline{1, N}$).

The values $x_0(s_l \mid \tau)$ and $\psi_0(s_l \mid \tau)$ ($l = \overline{1, N(\tau)}$) are calculated using the information obtained before the controller was turned on and after solving the basic problem:

$$x^0(s_l|\tau) = F(s_l)F^{-1}(\tau)x^*(\tau) + \sum_{j=0}^{j(l)-1} F(s_l)\int_{t_j^0(\tau)}^{t_{j+1}^0(\tau)} F^{-1}(\vartheta)bd\vartheta(-1)^j\gamma(\tau)$$

$$+ \int_{t_{j(l)}^0(\tau)}^{s_l} F(s_l)F^{-1}(\vartheta)bd\vartheta(-1)^{j(l)}\gamma(\tau) + \int_{\tau}^{s_l} F(s_l)F^{-1}(\vartheta)ad\vartheta,$$

$$\psi'_0(s_l|\tau) = (c' - \nu^{0\prime}(\tau)H)F(t^*)F^{-1}(s_l), \quad l = \overline{1, N(\tau)}.$$

Solving Eq. (3.12), we find $z_1(\tau)$ and calculate $t_j^1(\delta \mid \tau) = t_j^0(\tau) + \delta t_j^1(\tau)$. If

$$\tau < t_1^1(\delta|\tau) < \ldots < t_{k^*(\tau)}^1(\delta|\tau) < t^*, \tag{3.14}$$

then the structure of the control $u^1(t, \delta \mid \tau, x^*(\tau))$ ($t \in T^\tau$) is identical to the structure of the optimal control $u^0(t \mid \tau, x^*(\tau))$ ($t \in T^\tau$) of the basic problem. In this case, the control $u^{1*}(\tau) = \gamma(\tau)$ is fed to system (2.4). The violation of condition (3.14) means that δ is greater than a number $\mu_0 > 0$ (see [7]) such that, for $|\mu| < \mu_0$, the structure of the optimal control $u^0(t, \mu \mid \tau, x^*(\tau))$ ($t \in T^\tau$) is identical to the structure of $u^0(t \mid \tau, x^*(\tau))$ ($t \in T^\tau$). In this case, we set

$$u^{1*}(\tau) = \begin{cases} \gamma(\tau), & t_1^1(\delta|\tau) > \tau, \\ -\gamma(\tau), & t_1^1(\delta|\tau) < \tau. \end{cases}$$

In doing so, we assume that the lengths of the intervals at which the structures of the controls $u^1(t, \delta \mid \tau, x^*(\tau))$ ($t \in T^\tau$) and $u^0(t \mid \tau, x^*(\tau))$ ($t \in T^\tau$) differ from each other are insignificant compared with the duration t^* of the process.

Remark 1. The violation of condition (3.14) can manifest itself as follows: (1) $t_1^1(\delta|\tau) < \tau$; (2) $t_k^1(\delta|\tau) > t_{k+1}^1(\delta \mid \tau)$; (3) $t_{k^*(\tau)}^1(\delta|\tau) > t^*$. In such cases, the solution to the basic problem is called quasi-singular. To analyze the quasi-linearity, for example, in case (1), we first find μ_0 satisfying the equation $t_1^1(\mu_0 \mid \tau) = \tau$ from the asymptotic expansion. Then, we define the new quasilinear system

$$\dot{x} = Ax + a\mu_0 g(x) + \nu g(x) + bu, \quad x(\tau) = x^*(\tau), \quad \nu = \mu - \mu_0 \longrightarrow 0.$$

An asymptotic expansion in ν is constructed using the asymptotic solution to the nonlinear basic system (see [8])

$$\dot{x} = Ax + a + \mu_0 g(x) + bu, \quad x(\tau) = x^*(\tau).$$

Quasi-singularity of the solution to the basic problem can also manifest itself in that the optimal control $u^0(t, \delta \mid \tau, x^*(\tau))$ ($t \in T^\tau$) has switching points that are not taken into account in the structure of the control $u^0(t \mid \tau, x^*(\tau))$ ($t \in T^\tau$); for example, the $(k^*(\tau) + 1)$th switching point can be close to t^*. To analyze such situations, “imaginary” (which are beyond the interval T^τ) zeros of the cocontrol of the basic problem are used.

3.4. An Example

Consider the optimal control problem for a dynamical system driven by the Duffing equation

$$x_1(7) \longrightarrow \max, \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + x_1^3/6 + u, \quad x_1(0) = x_2(0) = 0,$$
$$x_2(7) = 0, \quad |u(t)| \le L, \quad t \in T = [0, 7]. \tag{3.15}$$

We use the following values of the control intensity L: $L_1 = 0.1$, $L_2 = 0.2$, $L_3 = 0.3$, and the corresponding sets $X_1 = \{(x_1, x_2) : |x_1| < \pi/6\}$, $X_2 = \{(x_1, x_2) : |x_1| < \pi/3\}$, $X_3 = \{(x_1, x_2) : |x_1| < \pi/2\}$. The system behavior will be analyzed for these sets. On every set, we approximate the nonlinear element of the system, $-x_1 + x_1^3/6$, by the function $-x_1$. The corresponding approximation error δ is $\delta_1 = 0.04788$, $\delta_2 = 0.22365$, and $\delta_3 = 0.698466$. Thus, problem (3.15) is considered as an element of the following family of problems on each of the sets X_i ($i = 1, 2, 3$):

$$x_1(7) \longrightarrow \max, \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + \mu x_1^3/6\delta_i + u, \quad x_1(0) = x_2(0) = 0,$$
$$x_2(7) = 0, \quad |u(t)| \le L_i, \quad t \in T, \quad i = 1, 2, 3. \tag{3.16}$$

Table 1 presents the results of an open loop solution to problem (3.15). Trajectories of system (3.15) were constructed for the following controls: (1) $u^0(t)$ ($t \in T$), which is the optimal control in the basic problem

$$x_1(7) \longrightarrow \max, \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u, \quad x_1(0) = x_2(0) = 0,$$
$$x_2(7) = 0, \quad |u(t)| \le L_i, \quad t \in T, \quad i = 1, 2, 3;$$

(2) $u_\delta^1(t)$ ($t \in T$), which is a realization of the asymptotically 1-optimal open loop control in problem (3.16) for the fixed $\mu = \delta_i$; 3) $u_\delta^2(t)$ ($t \in T$), which is a realization of the asymptotically 2-optimal open loop control in problem (3.16) for the fixed $\mu = \delta_i$; 4) $u^0(t, \delta)$ ($t \in T$), which is the optimal open loop control in the nonlinear problem (3.15) produced by the refinement procedure described in [7]. In each case, the control was of the form

$$u(t) = \begin{cases} L_i, & t \in [0, t_1[, \\ -L_i, & t \in [t_1, t_2[, \\ L_i, & t \in [t_2, 7[. \end{cases} \tag{3.17}$$

Table 1 presents the switching points of these controls an the terminal states of system (3.15): $x_1(7)$ is the value of the objective functional, and $x_2(7)$ is the terminal residual.

It is seen from Table 1 that, for small δ (the set X_1), the optimal control of the linear system yields satisfactory results (row 1) compared with the optimal control (row 4) in the nonlinear problem. An asymptotic correction of the first (row 2) and the second (row 3) order significantly improves the quality of control.

As δ increases (the set X_2), the quality of the optimal control in the linear problem sharply deteriorates (row 1, the set X_2), and the first-order correction (row 2) provides the same quality as the optimal control in the linear problem on the set X_1. This quality improves by an order of magnitude when the second-order correction is used (row 3).

If the set X is expanded, corrections are not that effective (see Table 1 for X_3). In this case, piecewise linear problems should be used as basic ones (see below).

Consider the construction of a closed loop solution for the case $L = L_3 = 0.3$. First, we analyze the behavior of system (3.15) driven by the optimal open loop control $u^0(t, \delta_3)$ ($t \in T$) and the realization $u^{1*}(t)$ ($t \in T$) constructed by the 1-optimal controller ($h = 0.01$) in the absence of perturbations. Solutions to Eqs. (3.13) were found by the Runge–Kutta–Felberg method of the fourth or fifth order [13]. The control $u^{1*}(t)$ ($t \in T$) has the form (3.17), and its switching points are 0.34, 3.49, and 6.908765. At $t^* = 7$, the trajectory of system (3.15) driven by the control $u^{1*}(t)$ ($t \in T$) reached the point (1.372104, 0.022574).

Let system (3.15) be affected by an unknown perturbation $w^*(t)$ ($t \in [0, t^0[$, $t^0 < 7$), $w^*(t) \equiv 0$ for $t \ge t^0$, and the system is governed by the equation

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + x_1^3/6 + u + w^*(t), \quad x_1(0) = x_2(0) = 0. \tag{3.18}$$

System (3.18) affected by various perturbations $w^*(t)$ ($t \in [0, t^0[$) was fed with the following realizations of the optimal feedback: (1) $u^{1*}(t)$ ($t \in T$) produced by the 1-optimal controller that solved Eqs. (3.13) by the method described in [13]; (2) $u_N^{1*}(t)$ ($t \in T$) produced by the 1-optimal controller that solved Eqs. (3.13) by the mean rectangular quadrature formula with N nodes. These controls were given by formula (3.17).

For every perturbation, Table 2 presents the switching points of the optimal feedback and the corresponding terminal states of system (3.18).

The results in Table 2 show that it is sufficient to use a relatively small number of nodes in the quadrature formula; i.e., the asymptotic correction procedure places moderate requirements upon the computer memory.

4. GLOBAL OPTIMIZATION OF NONLINEAR SYSTEMS

In this section, we consider optimization of the nonlinear system (2.1) on the set X for which the error δ of the approximation $\hat{f}(x)$ $(x \in X)$ does not ensure that the technique described in Section 3 yields satisfactory results.

The idea of extending this technique to larger sets is based on piecewise linear approximation of the nonlinear element of the system. As in [5], we assume that the closure of X can be represented as a union of polyhedral sets $X_1, \ldots, X_p$ such that $\operatorname{int}X_i \cap \operatorname{int}X_j = \varnothing$ for $i \neq j$. The approximation $\hat{f}(x)$ $(x \in X)$ is defined as a continuous function that is linear on each set X_j $(j = \overline{1, p})$. The approximation error δ is again defined by formula (3.1). Again, the solution to problem (2.1) involves two procedures: (1) solution of the piecewise linear problem; (2) asymptotic correction of the solution to the piecewise linear problem.

4.1. Solution of the Piecewise Linear Problem

In [5], an algorithm for the construction of an open loop and closed loop optimal controls for the piecewise linear system

$$c'x(t^*) \longrightarrow \max, \quad \dot{x} = \hat{f}(x) + bu, \quad x(0) = x_0, \quad x(t^*) \in X^*, \quad |u(t)| \le 1, \quad t \in T. \tag{4.1}$$

was described. Every feasible control for the problem under consideration generates a trajectory that goes through a sequence $X_{i_1}, \ldots, X_{i_k}$ of sets from $X_1, \ldots, X_p$ and crosses the boundaries of adjacent sets at instants $\theta_{i_1}, \ldots, \theta_{i_{k-1}} \in T$. Such sequences will be called feasible, and the sequence corresponding to the optimal control will be called optimal. Let $X^0 = \{X_1^0, \ldots, X_{j^*}^0\}$ be an optimal sequence of sets, and the affine hull of $X_j^0 \cap X_{j+1}^0$ be the variety $\{x \in \mathbb{R}^n : H_jx = g_j\}$, where H_j is an m_j-by-n matrix and g_j is an m_j-dimensional vector $(j = \overline{1, j^* - 1})$. The sequence X^0 is called the structure of the optimal trajectory for problem (4.1). The function $\hat{f}(x)$ has the form $\hat{f}(x) = A_jx + a_j$ on the set X_j^0, where $A_j \in \mathbb{R}^{n \times n}$ and $a_j \in \mathbb{R}^n$. Moreover, $A_jx + a_j = A_{j+1}x + a_{j+1}$ if $H_jx = g_j$ $(j = \overline{1, j^* - 1})$ and H_j $(j = \overline{1, j^* - 1})$ are such that $\operatorname{rank}\begin{pmatrix} A_j - A_{j+1} \\ H_j \end{pmatrix} < n$ for $j = \overline{1, j^* - 1}$. For convenience, we rename the variables as follows: $\theta_0 = 0$, $\theta_{j^*} = t^*$, $H_{j^*} = H$, $g_{j^*} = g$, and $m_{j^*} = m$.

Thus, if the structure X^0 of the optimal trajectory is known, problem (4.1) can be represented as an optimal control problem for the set of linear systems

$$J(\theta, u) = c'x(\theta_{j^*}) \longrightarrow \max, \tag{4.2}$$

$$\dot{x}(t) = A_jx(t) + a_j + bu(t), \quad t \in [\theta_{j-1}, \theta_j[, \quad j \in J = \{1, 2, \ldots, j^*\}, \quad x(\theta_0) = x_0, \tag{4.3}$$

$$H_jx(\theta_j) = g_j, \quad j \in J, \tag{4.4}$$

$$\begin{gathered} |u(t)| \le 1, \quad t \in T, \quad \theta_0 < \ldots < \theta_{j^*}, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}, \quad g_j \in \mathbb{R}^{m_j}, \\ j = \overline{1, j^*}, \quad \operatorname{rank}H_j = m_j < n, \quad \theta_j \in T_h, \quad j = \overline{1, j^* - 1}, \quad \theta_0 = 0, \quad \theta_{j^*} = t^*. \end{gathered} \tag{4.5}$$

In problem (4.2)–(4.5), the instants $\theta = (\theta_1, \ldots, \theta_{j^*-1})$ are calculated along with the control $u(t)$ $(t \in T)$.

A vector θ and a discrete control $u(\cdot)$ are called an available control for problem (4.2)–(4.5) if they satisfy constraints (4.5). An available control $\{\theta, u(\cdot)\}$ and the corresponding trajectory $x(t) = x(t \mid \theta, u(\cdot))$ $(t \in T)$ of system (4.3) are called feasible if $x(t)$ $(t \in T)$ satisfies constraints (4.4). A feasible control $\{\theta^0, u^0(\cdot)\}$ is called optimal if it maximizes the objective functional (4.2).

Problem (4.2)–(4.5) is solved in two steps.

Table 1

Set	Control ($t \in T$)	Switching points	Terminal state
X_1	$u^0(t)$	0.607097	0.424657
		3.748689	0.009744
		6.890282	
	$u_\delta^1(t)$	0.580273	0.423734
		3.726758	0.000219
		6.890355	
	$u_\delta^2(t)$	0.580227	0.423731
		3.726717	0.000199
		6.890336	
	$u^0(t, \delta)$	0.579609	0.423709
		3.726218	10^{-7}
		6.890434	
X_2	$u^0(t)$	0.607097	0.870482
		3.748689	0.079389
		6.890282	
	$u_\delta^1(t)$	0.499801	0.867089
		3.660964	0.007758
		6.890572	
	$u_\delta^2(t)$	0.489997	0.866479
		3.653117	0.001670
		6.890529	
	$u^0(t, \delta)$	0.486903	0.866301
		3.650663	10^{-6}
		6.892421	
X_3	$u^0(t)$	0.607097	1.353878
		3.748689	0.274193
		6.890282	
	$u_\delta^1(t)$	0.365681	1.372831
		3.551306	0.071367
		6.890934	
	$u_\delta^2(t)$	0.362827	1.372702
		3.548675	0.067883
		6.899304	
	$u^0(t, \delta)$	0.258456	1.370287
		3.468555	10^{-6}
		6.915103	

Table 2

$w^*(t)$, $t < t^\circ$	t°	Control ($t \in T$)	Switching points	Terminal state
$-0.3\sin(4t)$	6	$u^{1*}(t)$	0.41	1.311424
			3.553450	0.067352
		$u_{50}^{1*}(t)$	0.41	1.311424
			3.553420	0.067381
		$u_{300}^{1*}(t)$	0.41	1.311424
			3.553449	0.067352
0.1	3	$u^{1*}(t)$	0.35	1.143317
			3.754725	0.023658
			6.896317	
		$u_{10}^{1*}(t)$	0.36	1.143558
			3.545102	0.023688
			6.837803	
		$u_{100}^{1*}(t)$	0.35	1.143316
			3.552888	0.023656
			6.839616	
-0.1	3	$u^{1*}(t)$	0.39	1.648328
			3.407981	0.159169
			6.946894	
		$u_{10}^{1*}(t)$	0.4	1.646080
			3.397529	0.161189
			6.943723	
		$u_{100}^{1*}(t)$	0.39	1.648322
			3.408122	0.159111
			6.946697	

1. First, problem (4.2)–(4.5) is linearized along a feasible trajectory. Formally, the linearized problem retains the form (4.2)–(4.5), but the vector θ is fixed. This is a problem of controlling a time-dependent linear system subjected to intermediate state constraints. It is solved by the algorithm described in [4], which is based on a dynamic implementation of the adaptive linear programming method (this algorithm was briefly described in Section 3).

2. The solution to the linearized problem is corrected by choosing optimal instants of transition from one domain of linearity to another.

A detailed description of this algorithm for solving problem (4.2)–(4.5) can be found in [5].

4.2. Asymptotic Correction of the Solution to the Piecewise Linear Problem

As in Section 3, we correct the solution to the piecewise linear problem using asymptotic expansions of the basic elements of the optimal control. To this end, we embed problem (2.1) in the following family of problems depending on a small parameter $\mu \longrightarrow 0$:

$$c'x(t^*) \longrightarrow \max, \quad \dot{x} = \hat{f}(x) + \mu g(x) + bu, \quad x(0) = x_0, \tag{4.6}$$

$$x(t^*) \in X^*, \quad |u(t)| \le 1, \quad t \in T,$$

where $g(x) = (f(x) - \hat{f}(x))/\delta$ $(x \in X)$. Taking into account the form of the function $\hat{f}(x)$ $(x \in X)$ and assuming that the structure X^0 of the optimal trajectory is known, we write problem (4.6) in a parametric form as an optimal control problem for the step-like quasilinear system

$$J(\theta, u) = c'x(\theta_{j*}) \longrightarrow \max, \tag{4.7}$$

$$\dot{x} = A_j x + a_j + \mu g_j(x) + bu, \quad t \in [\theta_{j-1}, \theta_j[, \quad j = \overline{1, j*}, \quad x(\theta_0) = x_0, \tag{4.8}$$

$$H_j x(\theta_j) = g_j, \quad j = \overline{1, j*}, \quad |u(t)| \le 1, \quad t \in T. \tag{4.9}$$

Here $g_j(x) = (f(x) - A_j x - a_j)/\delta$ and $x \in X_j^0$ $(j = \overline{1, j*})$. The functions $g_j(x)$ $(j = \overline{1, j*})$ are, obviously, infinitely differentiable in $\mathrm{int} X_j$, and $g_j(x) = g_{j+1}(x)$ if $H_j x = g_j$ for $j = \overline{1, j* - 1}$.

The vector $\theta(\mu) = (\theta_j(\mu), j = \overline{1, j* - 1})$ and the piecewise continuous function $u(t, \mu)$ $(t \in T)$ are referred to as an available control if $|u(t, \mu)| \le 1$ for $t \in T$. At the points of discontinuity of $u(t, \mu)$ $(t \in T)$, we assume it to be continuous from the right. An available control $\{\theta(\mu); u(t, \mu), t \in T\}$ is called feasible if the generated trajectory $x(t, \mu)$ $(t \in T)$ satisfies conditions (4.9) (by virtue of (4.8), this trajectory is continuous). A feasible control is called optimal if it maximizes the objective functional (4.7).

Now, we describe an algorithm for the asymptotic solution of problem (4.7)–(4.9). First, we give a precise definition of the asymptotic approximation to the optimal control.

A family of available controls $\{\theta_\mu^s; u_\mu^s(t), t \in T; \mu \longrightarrow 0\}$ is called an asymptotically optimal control of order s (asymptotically s-optimal for $s = 0, 1, \ldots$) if the following conditions are satisfied for the trajectories $x_\mu^s(t)$ $(t \in T, \mu \longrightarrow 0)$ of system (4.8) generated by the controls $\{\theta_\mu^s, u_\mu^s(t), t \in T\}$ $(\mu \longrightarrow 0)$:

$$c'x_\mu^s(\theta_{j*}) - c'x^0(\theta_{j*}, \mu) = O(\mu^{s+1}), \quad H_j x_\mu^s(\theta_{\mu j}^s) - g_j = O_j(\mu^{s+1}), \quad j = \overline{1, j*}.$$

Here, $x^0(t, \mu)$ $(t \in T)$ is the trajectory of system (4.8) generated by the optimal control $\{\theta^0(\mu); u^0(t, \mu), t \in T\}$.

Below, we propose an algorithm that, for any given positive integer s, produces an asymptotically s-optimal control for problem (4.7)–(4.9). Basically, this algorithm is close to that designed in [6, 7] for the asymptotic solution of the quasilinear terminal control problem. It is based on the construction of asymptotic expansions in integer powers of the small parameter of the vector $\theta^0(\mu)$ and of the switching points of the relay control $u^0(t, \mu)$ $(t \in T)$.

Moreover, we show how the asymptotic approximations to the optimal control can be used to exactly solve problem (4.7)–(4.9) for a given value of the small parameter.

4.3. Analysis of the Basic Problem

Problem (4.2)–(4.5) is formally obtained from problem (4.7)–(4.9) by setting $\mu = 0$. It is called the basic problem.

Assumption 1. Problem (4.2)–(4.5) has a solution $\{\theta^0 = (\theta_j^0, j = \overline{1, j* - 1}); u^0(t), t \in T\}$.

By the maximum principle [14], there exists a number $\lambda_0 \ge 0$ and vectors $\nu_1^0, \ldots, \nu_{j*}^0$ of the dimensions $m_1, \ldots, m_{j*}$, respectively, such that

$$\psi^{0\prime}(t) b u^0(t) = \max_{|u| \le 1} \psi^{0\prime}(t) b u, \quad t \in T, \tag{4.10}$$

where $\psi^0(t)$ $(t \in T)$ is a piecewise smooth vector function that solves the adjoint system

$$\dot{\psi} = -A_j' \psi, \quad t \in]\theta_{j-1}^0, \theta_j^0], \quad j = \overline{1, j*}, \quad (\theta_0^0 = \theta_0, \theta_{j*}^0 = \theta_{j*}),$$

$$\psi(\theta_{j*}) = \lambda_0 c - H_{j*}' \nu_{j*}^0$$

and has discontinuities at the points $\theta_1^0, \ldots, \theta_{j*-1}^0$, where it is continuous from the right, and

$$\psi(\theta_j^0 - 0) = \psi(\theta_j^0) - H_j' \nu_j^0, \quad j = \overline{1, j* - 1}.$$

In addition, the following conditions are fulfilled:

$$\psi^{0\prime}(\theta_j^0)b(u^0(\theta_j^0-0)-u^0(\theta_j^0)) = \nu_j^{0\prime}H_j(A_jx^0(\theta_j^0)+a_j+bu^0(\theta_j^0-0)), \quad j = \overline{1, j^*-1}. \tag{4.11}$$

The trajectory generated by the optimal control $\{\theta^0; u^0(t), t \in T\}$ is denoted by $x^0(t)$ $(t \in T)$.

Assumption 2. $\lambda_0 > 0$.

Then, without loss of generality, we can assume that $\lambda_0 = 1$.

Assumption 3. The cocontrol $\Delta^0(t) = \psi^{0\prime}(t)b$ $(t \in T)$ has a finite number of zeros $t_1^0, \ldots, t_{k^*}^0$, which do not include the points $\theta_0, \ldots, \theta_{j^*}$, and the conditions $\Delta^0(\theta_j^0 - 0) \neq 0$ $(j = \overline{1, j^*-1})$ and $\dot{\Delta}^0(t_j^0) \neq 0$ for $j = \overline{1, k^*}$.

It is supposed that the zeros of the cocontrol are enumerated in ascending order. Under our assumption, they will be switching points of the function $u^0(t)$ $(t \in T)$, which is a relay function due to (4.10): $u^0(t) = \operatorname{sgn}\Delta^0(t)$ $(t \in T)$. In addition to $t_1^0, \ldots, t_{k^*}^0$, only the instants $\theta_1^0, \ldots, \theta_{j^*-1}^0$ can be used as switching points.

Assume that there are k_j zeros of the cocontrol in the interval $[\theta_0, \theta_j^0[$ $(j = \overline{1, j^*-1})$. Consider the numbers $\gamma_1 = \operatorname{sgn}\Delta^0(\theta_0)$, $\gamma_j = \operatorname{sgn}\Delta^0(\theta_{j-1}^0)$ $(j = \overline{2, j^*})$ and the vectors $T_j^0 = (t_i^0, i = \overline{k_{j-1}+1, k_j})$ for $j = \overline{1, j^*}$, where $k_0 = 0$ and $k_{j^*} = k^*$.

Upon solving the basic problem, we form the matrix

$$I_0 = \begin{bmatrix} B_1 & B_2 & 0 \\ B_3 & B_4 & B_5 \\ B_6 & B_7 & B_8 \end{bmatrix}. \tag{4.12}$$

Here, $B_1 = (B_{jl}^{(1)}, j, l = \overline{1, j^*})$ and $B_{jl}^{(1)} = (2(-1)^{i-k_{l-1}+1}\gamma_l H_j \Phi_{jl}\varphi_l(t_i^0), i = \overline{k_{l-1}+1, k_l})$, where

$$\Phi_{jl} = \begin{cases} \prod_{r=l}^{j-1} F_{r+1}(\theta_r^0), & j > l, \\ E, & j = l, \\ 0, & j < l, \end{cases}$$

$\varphi_l(t) = F_l(t)b$ $(l = \overline{1, j^*})$, and $F_l(t)$ is an n-by-n matrix function that solves the initial problem $\dot{F}_l = -F_lA_l$, $F_l(\theta_l) = E$. The other blocks of matrix (4.12) are as follows:

$$B_2 = (B_{jl}^{(2)}, j = \overline{1, j^*}, l = \overline{1, j^*-1});$$

$$B_{jl}^{(2)} = H_j\Phi_{jl}(A_l - A_{l+1})\left[\Phi_{l0}x_0 + \sum_{i=1}^{l}\Phi_{li}\int_{\theta_{i-1}}^{\theta_i}F_i(\vartheta)(bu^0(\vartheta)+a_i)d\vartheta\right]$$

$$+ H_j\Phi_{jl}[(\gamma_l(-1)^{k_l-k_{l-1}} - \gamma_{l+1})b + a_l - a_{l+1}], \quad j \neq l;$$

$$B_{jj}^{(2)} = H_j(A_jx^0(\theta_j^0) + a_j + \gamma_j(-1)^{k_j-k_{k-1}}b);$$

$$B_3 = \operatorname{diag}(\dot{\Delta}^0(t_i^0), i = \overline{1, k^*});$$

$$B_4 = (B_{jl}^{(4)}, j = \overline{1, j^*}, l = \overline{1, j^*-1}), \quad B_{jl}^{(4)} = ((\psi^{0\prime}(\theta_l^0-0)A_l - \psi^{0\prime}(\theta_l^0)A_{l+1})\Phi_{lj}\varphi_j(t_j^0), i = \overline{k_{j-1}+1, k_j})';$$

$$B_5 = (B_{jl}^{(5)}, j, l = \overline{1, j^*}), \quad B_{jl}^{(5)} = -(H_l\Phi_{lj}\varphi_j(t_j^0), i = \overline{k_{j-1}+1, k_j})';$$

$$B_6 = (B^{(6)}_{jl},\ j = \overline{1, j^*-1},\ l = \overline{1, j^*}), \quad B^{(6)}_{jl} = (2(-1)^{i-k_{l-1}+1}\gamma_l \nu^{0\prime}_j H_j A_j \Phi_{jl}\varphi_l(t^0_i),\ i = \overline{k_{l-1}+1, k_l});$$

$$B_7 = (B^{(7)}_{jl},\ j, l = \overline{1, j^*-1}),$$

$$B^{(7)}_{jl} = \nu^{0\prime}_j H_j \Phi_{jl}(A_l - A_{l+1})\left[\Phi_{l0}x_0 + \sum_{i=1}^{l}\Phi_{li}\int_{\theta_{i-1}}^{\theta_i} F_i(\vartheta)(bu^0(\vartheta) + a_i)d\vartheta\right]$$

$$+ \nu^{0\prime}_j H_j \Phi_{jl}[(\gamma_l(-1)^{k_l-k_{l-1}} - \gamma_{l+1})b + a_l - a_{l+1}] + (\gamma_{j+1} - \gamma_j(-1)^{k_j-k_{j-1}})(\psi^{0\prime}(\theta^0_l - 0)A_l - \psi^{0\prime}(\theta^0_l)A_{l+1})\Phi_{lj}b,$$

$$j \neq l, \quad B^{(7)}_{jj} = \nu^{0\prime}_j H_j A_j(A_j x^0(\theta^0_j) + a_j + \gamma_j(-1)^{k_j-k_{j-1}}b) + (\gamma_j(-1)^{k_j-k_{j-1}} - \gamma_{j+1})\psi^{0\prime}(\theta^0_j)A_{j+1}b;$$

$$B_8 = (B^{(8)}_{jl},\ j = \overline{1, j^*-1},\ l = \overline{1, j^*}), \quad B^{(8)}_{jl} = ((\gamma_j(-1)^{k_j-k_{j-1}} - \gamma_{j+1})H_l\Phi_{lj}b)', \quad j \neq l;$$

$$B^{(8)}_{jj} = (H_j(A_j x^0(\theta^0_j) + a_j + \gamma_j(-1)^{k_j-k_{j-1}}b))'.$$

Assumption 4. The conditions $\operatorname{rank} B_1 = \sum_{j=1}^{j^*} m_j$ and $\det I_0 \neq 0$ are fulfilled.

4.4. The Structure and Asymptotic Behavior of the Solution to Problem (4.7)–(4.9)

Before proceeding with the description of the algorithm for constructing asymptotically optimal controls for problem (4.7)–(4.9), we prove a theorem that establishes the structure and the asymptotic properties of the solution to this problem. The proof is constructive and, thus, predefines further calculations.

Let $t_1, \ldots, t_{k_1}, \theta_1, t_{k_1+1}, \ldots, t_{k_2}, \theta_2, \ldots, \theta_{j^*-1}, t_{k_{j^*-1}+1}, \ldots, t_{k^*}$ be different numbers from the interval $]\theta_0, \theta_{j^*}[$ ordered in ascending order. Consider the vectors $\theta = (\theta_1, \ldots, \theta_{j^*-1})$ and $T_j = (t_i, i = \overline{k_{j-1}+1, k_j})$ ($j = \overline{1, j^*}$), and denote by $x(t, T_1, \ldots, T_{j^*}, \theta, \mu)$ ($t \in T$) the trajectory of system (4.8) generated by the available control $\{\theta; u(t, T_1, \ldots, T_{j^*}, \theta), t \in T\}$, where

$$u(t, T_1, \ldots, T_{j^*}, \theta) = \begin{cases} \gamma_1, \quad t \in [\theta_0, t_1[, \\ (-1)^i\gamma_1, \quad t \in [t_i, t_{i+1}[, \quad i = \overline{1, k_1-1}, \\ (-1)^{k_1}\gamma_1, \quad t \in [t_{k_1}, \theta_1[, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \gamma_{j^*}, \quad t \in [\theta_{j^*-1}, t_{k_{j^*-1}+1}[, \\ (-1)^{i-k_{j^*-1}}\gamma_{j^*}, \quad t \in [t_i, t_{i+1}[, \quad i = \overline{k_{j^*-1}+1, k^*-1}, \\ (-1)^{k_{j^*}-k_{j^*-1}}\gamma_{j^*}, \quad t \in [t_{k^*}, \theta_{j^*}[. \end{cases} \tag{4.13}$$

Furthermore, let $\nu_1, \ldots, \nu_{j^*}$ be vectors of the dimension $m_1, \ldots, m_{j^*}$, respectively. Denote by $\psi(t, T_1, \ldots, T_{j^*}, \theta, \nu_1, \ldots, \nu_{j^*}, \mu)$ ($t \in T$) the piecewise smooth vector function that solves the adjoint system

$$\dot{\psi} = -\left(A_j + \mu\frac{\partial g_j(x(t, T_1, \ldots, T_{j^*}, \theta, \mu))}{\partial x}\right)'\psi, \quad t \in]\theta_{j-1}, \theta_j], \quad j = \overline{1, j^*},$$
$$\psi(\theta_{j^*}) = c - H'_{j^*}\nu_{j^*} \tag{4.14}$$

and has discontinuities at the points $\theta_1, \ldots, \theta_{j^*-1}$, where it is continuous from the right, and

$$\psi(\theta_j - 0) = \psi(\theta_j) - H'_j\nu_j, \quad j = \overline{1, j^*-1}. \tag{4.15}$$

Theorem. *Under Assumptions* 1–4, *there exists a unique optimal control* $\{\theta(\mu); u^0(t, \mu), t \in T\}$ *in prob-*

lem (4.7)–(4.9) *with a sufficiently small (in absolute magnitude)* μ, *where*

$$u^0(t, \mu) = u(t, T_1(\mu), \dots, T_{j^*}(\mu), \theta(\mu)), \quad t \in T. \tag{4.16}$$

Moreover, the vectors $T_1(\mu), \dots, T_{j^*}(\mu), \theta(\mu)$ *are infinitely differentiable functions of the small parameter* μ and $T_j(0) = T_j^0$ $(j = \overline{1, j^*})$, $\theta(0) = \theta^0$.

Proof. Consider the system of equations

$$\begin{gathered} H_j x(\theta_j, T_1, \dots, T_{j^*}, \theta, \mu) - g_j = 0, \quad j = \overline{1, j^*}, \\ \psi'(t_i, T_1, \dots, T_{j^*}, \theta, \nu_1, \dots, \nu_{j^*}, \mu) b = 0, \quad i = \overline{1, k^*}, \\ \nu_j' H_j \{A_j x(\theta_j, T_1, \dots, T_{j^*}, \theta, \mu) + a_j + \mu g_j[x(\theta_j, T_1, \dots, T_{j^*}, \theta, \mu)] + (-1)^{k_j - k_{j-1}} \gamma_j b\} \\ + [\gamma_{j+1} - (-1)^{k_j - k_{j-1}} \gamma_j] \psi'(\theta_j, T_1, \dots, T_{j^*}, \theta, \nu_1, \dots, \nu_{j^*}, \mu) b = 0, \quad j = \overline{1, j^* - 1}, \end{gathered} \tag{4.17}$$

in the unknowns $T_1, \dots, T_{j^*}, \theta, \nu_1, \dots, \nu_{j^*}$ and verify, using the implicit function theorem, that it has a solution for sufficiently small μ.

Consider the vectors $z = (T_1, \dots, T_{j^*}, \theta, \nu_1, \dots, \nu_{j^*})$ and $z_0 = (T_1^0, \dots, T_{j^*}^0, \theta^0, \nu_1^0, \dots, \nu_{j^*}^0)$. Denote by $R(z, \mu)$ the vector function composed of the left-hand sides of Eqs. (4.17) and write this system in the form

$$R(z, \mu) = 0. \tag{4.18}$$

$R(z, \mu)$ is defined in the domain $\|z - z_0\| < \varepsilon$, $|\mu| < \mu_0$, where ε and μ_0 are sufficiently small positive numbers. Under the smoothness assumptions for the functions $g_j(x)$ $(j = \overline{1, j^*})$, it is infinitely differentiable. This can be verified by sequentially applying the theorem on the differentiability of solutions to ordinary differential equations with respect to the initial data and parameters to the primal system (4.8) and dual system (4.14), (4.15) on the intervals of constancy of function (4.13).

Note that $u(t, T_1^0, \dots, T_{j^*}^0, \theta^0) = u^0(t)$, $x(t, T_1^0, \dots, T_{j^*}^0, \theta^0, 0) = x^0(t)$, and $\psi(t, T_1, \dots, T_{j^*}, \theta^0, \nu_1^0 \dots, \nu_{j^*}^0) = \psi^0(t)$ for $t \in T$. Then, $H_j x(\theta_j^0, T_1^0, \dots, T_{j^*}^0, \theta^0, 0) - g_j = H_j x^0(\theta_j^0) - g_j$ for $j = \overline{1, j^*}$ and $\psi'(t_i^0, T_1^0, \dots, T_{j^*}^0, \theta^0, \nu_1^0, \dots, \nu_{j^*}^0, 0) b = \Delta^0(t_i^0) = 0$ for $i = \overline{1, k^*}$. Finally, by virtue of (4.11), we have $\nu_j^{0\prime} H_j[A_j x(\theta_j^0, T_1^0, \dots, T_{j^*}^0, \theta^0, 0) + a_j + (-1)^{k_j - k_{j-1}} \gamma_j b] + [\gamma_{j+1} - (-1)^{k_j - k_{j-1}} \gamma_j] \psi'(\theta_j^0, T_1^0, \dots, T_{j^*}^0, \theta^0, \nu_1^0, \dots, \nu_{j^*}^0, 0) b = \nu_j^{0\prime} H_j[A_j x^0(\theta_j^0) + a_j + b u^0(\theta_j^0 - 0)] + \psi^{0\prime}(\theta_j^0) b[u^0(\theta_j^0) - u^0(\theta_j^0 - 0)] = 0$ for $j = \overline{1, j^* - 1}$. Therefore, $R(z_0, 0) = 0$.

The Jacobian $\partial R(z_0, 0)/\partial z$ of system (4.17) is identical to I_0 (see (4.12)) and is, therefore, nonsingular.

Thus, system (4.18) or, which is the same, (4.17) satisfies the conditions of the implicit function theorem. By this theorem, infinitely differentiable functions $t_i(\mu)$ $(i = \overline{1, k^*})$, $\theta(\mu) = (\theta_1(\mu), \dots, \theta_{j^*-1}(\mu))$, and $\nu_j(\mu)$ $(j = \overline{1, j^*})$ are uniquely defined in a certain neighborhood of zero $|\mu| < \mu_1$ such that they satisfy Eqs. (4.17) and the conditions $t_i(0) = t_i^0$ $(i = \overline{1, k^*})$, $\theta(0) = \theta^0$, and $\nu_j(0) = \nu_j^0$ $(j = \overline{1, j^*})$. This means that, in problem (4.8) with a sufficiently small μ, there exists a feasible control $\{\theta(\mu); u^0(t, \mu), t \in T\}$ and vectors $\nu_1(\mu), \dots, \nu_{j^*}(\mu)$ such that the function $u^0(t, \mu)$ $(t \in T)$ has the form (4.16) and its switching points $t_i(\mu)$ $(i = \overline{1, k^*})$ are zeros of the cocontrol $\Delta(t, \mu) = \psi'(t, \mu) b$ $(t \in T)$ constructed on the basis of the piecewise smooth solution $\psi(t, \mu)$ $(t \in T)$ to the adjoint system

$$\dot{\psi} = -\left(A_j + \mu \frac{\partial g_j(x^0(t, \mu))}{\partial x}\right)' \psi, \quad t \in]\theta_{j-1}(\mu), \theta_j(\mu)], \quad j = \overline{1, j^*},$$

$$\theta_0(\mu) = \theta_0, \quad \theta_{j^*}(\mu) = \theta_{j^*},$$

$$\psi(\theta_{j^*}) = c - H_{j^*}' \nu_{j^*}(\mu), \quad \psi(\theta_j(\mu) - 0, \mu) = \psi(\theta_j(\mu), \mu) - H_j' \nu_j(\mu), \quad j = \overline{1, j^* - 1},$$

where $x^0(t, \mu)$ $(t \in T)$ is the trajectory of system (4.8) generated by the control $\{\theta(\mu); u^0(t, \mu), t \in T\}$. In addition, the conditions

$$\begin{aligned} &\psi'(\theta_j(\mu), \mu)[A_{j+1}x^0(\theta_j(\mu), \mu) + a_{j+1} + bu^0(\theta_j(\mu), \mu)] \\ &= \psi'(\theta_j(\mu) - 0, \mu)[A_j x^0(\theta_j(\mu), \mu) + a_j + bu^0(\theta_j(\mu) - 0, \mu)], \quad j = \overline{1, j^* - 1} \end{aligned} \tag{4.19}$$

are fulfilled. Using Assumption 3 and the implicit function theorem (more precisely, the assertion about the unique solvability of the equations), it is easy to show that, for a sufficiently small μ, the function $\Delta(t, \mu)$ $(t \in T)$ vanishes only at the points $t_i(\mu)$ $(i = \overline{1, k^*})$, and $u^0(t, \mu) = \operatorname{sgn}\Delta(t, \mu)$ $(t \in T)$. In combination with (4.19), this means that the feasible control $\{\theta(\mu); u^0(t, \mu), t \in T\}$ satisfies the maximum principle [14]. To complete the proof of the theorem, it remains to verify that the control $\{\theta(\mu); u^0(t, \mu), t \in T\}$ is optimal and that problem (4.8) has no other solutions. This can be done using a reasoning similar to that used in the proof of the fundamental theorem in Section 2.1 in [7].

4.5. Construction of the Asymptotic Expansion

Since the vector functions $\theta(\mu), T_1(\mu), \ldots, T_{j^*}(\mu)$ are infinitely differentiable, the following asymptotic expansions are valid:

$$\theta(\mu) \sim \theta^0 + \sum_{k=1}^{\infty} \mu^k \theta^k, \quad T_j(\mu) \sim T_j^0 + \sum_{k=1}^{\infty} \mu^k T_j^k, \quad j = \overline{1, j^*}.$$

Let a positive integer s be given. It is easy to verify that the family of controls $\{\theta^s(\mu); u^s(t, \mu), t \in T; \mu \longrightarrow 0\}$, where

$$u^s(t, \mu) = u(t, T_1^s(\mu), \ldots, T_{j^*}^s(\mu), \theta^s(\mu)), \quad t \in T, \tag{4.20}$$

$$\theta^s(\mu) \sim \theta^0 + \sum_{k=1}^{s} \mu^k \theta^k, \quad T_j^s(\mu) \sim T_j^0 + \sum_{k=1}^{s} \mu^k T_j^k, \quad j = \overline{1, j^*}, \tag{4.21}$$

is an asymptotically s-optimal control in problem (4.8). To construct this control, one must find the coefficients $\theta^k, T_1^k, \ldots, T_{j^*}^k$ $(k = \overline{1, s})$ of Taylor's polynomials (4.21). Here is an algorithm for the calculation of those coefficients. First of all, we expand the left-hand sides of Eqs. (4.17) in powers of the small parameter.

The vector functions $x(t, T_1, \ldots, T_{j^*}, \theta, \mu)$ and $\psi(t, T_1, \ldots, T_{j^*}, \theta, \nu_1, \ldots, \nu_{j^*}, \mu)$ $(t \in T)$ are infinitely differentiable at every point of their domain and, therefore, admit the asymptotic expansion

$$\begin{aligned} x(t, T_1, \ldots, T_{j^*}, \theta, \mu) &\sim \sum_{k=0}^{\infty} \mu^k x_k(t, T_1, \ldots, T_{j^*}, \theta), \\ \psi(t, T_1, \ldots, T_{j^*}, \theta, \nu_1, \ldots, \nu_{j^*}, \mu) &\sim \sum_{k=0}^{\infty} \mu^k \psi_k(t, T_1, \ldots, T_{j^*}, \theta, \nu_1, \ldots, \nu_{j^*}). \end{aligned} \tag{4.22}$$

Using the Poincaré formalism, we set up the following differential equations for the functions $x_k(t) = x_k(t, T_1, \ldots, T_{j^*}, \theta)$ and $\psi_k(t) = \psi_k(t, T_1, \ldots, T_{j^*}, \theta, \nu_1, \ldots, \nu_{j^*})$ $(k = \overline{0, s})$:

$$\begin{aligned} &\dot{x}_0 = A_j x_0 + a_j + bu(t, T_1, \ldots, T_{j^*}, \theta), \quad t \in [\theta_{j-1}, \theta_j[, \quad j = \overline{1, j^*}, \quad x_0(\theta_0) = x_0, \\ &\dot{\psi}_0 = -A_j'\psi_0, \quad t \in]\theta_{j-1}, \theta_j], \quad j = \overline{1, j^*}, \quad \psi_0(\theta_{j^*}) = c - H_{j^*}'\nu_{j^*}, \\ &\psi_0(\theta_j - 0) = \psi_0(\theta_j) - H_j'\nu_j, \quad j = \overline{1, j^* - 1}, \end{aligned} \tag{4.23a}$$

$$\begin{aligned} &\dot{x}_1 = A_j x_1 + g_j(x_0(t)), \quad t \in [\theta_{j-1}, \theta_j[, \quad j = \overline{1, j^*}, \quad x_1(\theta_0) = 0, \\ &\dot{\psi}_1 = -A_j'\psi_1 - \frac{\partial H_j(x_0(t), \psi_0(t))}{\partial x}, \quad t \in]\theta_{j-1}, \theta_j], \quad j = \overline{1, j^*}, \quad \psi_1(\theta_{j^*}) = 0, \end{aligned} \tag{4.23b}$$

$$\dot{x}_2 = A_j x_2 + \frac{\partial g_j(x_0(t))}{\partial x} x_1(t), \quad t \in [\theta_{j-1}, \theta_j[, \quad j = \overline{1, j^*}, \quad x_2(\theta_0) = 0,$$

$$\dot{\psi}_2 = -A_j'\psi_2 - \frac{\partial H_j(x_0(t), \psi_1(t))}{\partial x} - \frac{\partial^2 H_j(x_0(t), \psi_0(t))}{\partial x^2} x_1(t), \quad t \in]\theta_{j-1}, \theta_j], \quad j = \overline{1, j^*},$$

$$\psi_2(\theta_{j^*}) = 0,$$

....

where $H_j(x, \psi) = \psi' g_j(x)$ ($j = \overline{1, j^*}$). Note that the functions $\psi_k(t)$ and ($t \in T$, $k = \overline{1, s}$) are continuous, as well as the functions $x_k(t)$ ($t \in T$, $k = \overline{0, s}$).

By virtue of (4.22), the following asymptotic expansion takes place:

$$R(z, \mu) \sim \sum_{k=0}^{\infty} \mu^k R_k(z).$$

Here,

$$R_0(z) = \begin{bmatrix} H_j x_0(\theta_j, T_1, \ldots, T_{j^*}, \theta) - g_j, \quad j = \overline{1, j^*} \\ \psi_0'(t_i, T_1, \ldots, T_{j^*}, \theta, \nu_1, \ldots, \nu_{j^*})b, \quad i = \overline{1, k^*} \\ \nu_j' H_j[A_j x_0(\theta_j, T_1, \ldots, T_{j^*}, \theta) + a_j + (-1)^{k_j - k_{j-1}}\gamma_j b] \\ + [\gamma_{j+1} - (-1)^{k_j - k_{j-1}}\gamma_j]\psi_0'(\theta_j, T_1, \ldots, T_{j^*}, \theta, \nu_1, \ldots, \nu_{j^*})b, \quad j = \overline{1, j^* - 1} \end{bmatrix},$$

$$R_1(z) = \begin{bmatrix} H_j x_1(\theta_j, T_1, \ldots, T_{j^*}, \theta), \quad j = \overline{1, j^*} \\ \psi_1'(t_i, T_1, \ldots, T_{j^*}, \theta, \nu_1, \ldots, \nu_{j^*})b, \quad i = \overline{1, k^*} \\ \nu_j' H_j[A_j x_1(\theta_j, T_1, \ldots, T_{j^*}, \theta) + g_j(x_0(\theta_j, T_1, \ldots, T_{j^*}, \theta))] \\ + [\gamma_{j+1} - (-1)^{k_j - k_{j-1}}\gamma_j]\psi_1'(\theta_j, T_1, \ldots, T_{j^*}, \theta, \nu_1, \ldots, \nu_{j^*})b, \quad j = \overline{1, j^* - 1} \end{bmatrix},$$

$$R_2(z) = \begin{bmatrix} H_j x_2(\theta_j, T_1, \ldots, T_{j^*}, \theta), \quad j = \overline{1, j^*} \\ \psi_2'(t_i, T_1, \ldots, T_{j^*}, \theta, \nu_1, \ldots, \nu_{j^*})b, \quad i = \overline{1, k^*} \\ \nu_j' H_j\{A_j x_2(\theta_j, T_1, \ldots, T_{j^*}, \theta) + [\partial g_j(x_0(\theta_j, T_1, \ldots, T_{j^*}, \theta))/\partial x] x_1(\theta_j, T_1, \ldots, T_{j^*}, \theta)\} \\ + [\gamma_{j+1} - (-1)^{k_j - k_{j-1}}\gamma_j]\psi_2'(\theta_j, T_1, \ldots, T_{j^*}, \theta, \nu_1, \ldots, \nu_{j^*})b, \quad j = 1, 2, \ldots, j^* - 1 \end{bmatrix},$$

....

Since the vector functions $\nu_1(\mu), \ldots, \nu_{j^*}(\mu)$ are infinitely differentiable, the following asymptotic expansions are valid:

$$\nu_j(\mu) \sim \nu_j^0 + \sum_{k=0}^{\infty} \mu^k \nu_j^k, \quad j = \overline{1, j^*}.$$

Let $z_k = (T_1^k, \ldots, T_{j^*}^k, \theta^k, \nu_1^k, \ldots, \nu_{j^*}^k)$ ($k = \overline{1, s}$) and $z_s(\mu) = \sum_{k=0}^{s} \mu^k z_k$.

Expand the vector function $\sum_{k=0}^{s} \mu^k R_k(z_s(\mu))$ in Taylor's series in powers of μ up to the order s inclusive and equate the coefficients (beginning with the one of μ) to zero. This yields nonsingular systems of linear equations

$$I_0 z_1 = -R_1(z_0),$$

$$I_0 z_2 = -\frac{\partial R_1}{\partial z}(z_0)z_1 - \frac{1}{2}z_1'\frac{\partial^2 R_0}{\partial z^2}(z_0)z_1 - R_2(z_0), \tag{4.24}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solving them one after the other, we obtain the vectors $T_1^k, \ldots, T_{j^*}^k, \theta^k, \nu_1^k, \ldots, \nu_{j^*}^k$ $(k = \overline{1, s})$. The Jacobian I_0 has the form (4.12). The right-hand sides of systems (4.24) are formed by integrating Eqs. (4.23) and solving the differential equations for the partial derivatives of the vector functions x_k and ψ_k with respect to the components of the vectors $T_1, \ldots, T_{j^*}, \theta, \nu_1, \ldots, \nu_{j^*}$ (see, e.g., [15]). It must be taken into account that $u(t, T_1^0, \ldots, T_{j^*}^0, \theta^0) = u^0(t)$, $x_0(t, T_1^0, \ldots, T_{j^*}^0, \theta^0) = x^0(t)$, $\psi(t, T_1^0, \ldots, T_{j^*}^0, \theta^0, \nu_1^0, \ldots, \nu_{j^*}^0) = \psi^0(t)$ for $t \in T$. In particular,

$$R_1(z_0) = \begin{bmatrix} H_j x_1^0(\theta_j^0), \quad j = \overline{1, j^*} \\ \psi_1^{0\prime}(t_i^0)b, \quad i = \overline{1, k^*} \\ \nu_1^{0\prime}H_j[A_j x_1^0(\theta_j^0) + g_j(x^0(\theta_j^0))] + [\gamma_{j+1} - (-1)^{k_j - k_{j-1}}\gamma_j]\psi_1^{0\prime}(\theta_j^0)b, \quad j = \overline{1, j^* - 1} \end{bmatrix},$$

where $x_1^0(t)$, $\psi_1^0(t)$ $(t \in T)$ are continuous solutions to the following systems of differential equations:

$$\dot{x}_1 = A_j x_1 + g_j(x^0(t)), \quad t \in [\theta_{j-1}^0, \theta_j^0[, \quad j = \overline{1, j^*}, \quad x_1(\theta_0) = 0,$$

$$\dot{\psi}_1 = -A_j'\psi_1 - \frac{\partial H_j(x^0(t), \psi^0(t))}{\partial x}, \quad t \in]\theta_{j-1}^0, \theta_j^0], \quad j = \overline{1, j^*}, \quad \psi_1(\theta_{j^*}) = 0.$$

Since the Jacobian I_0 has the form (4.12), the first system in (4.24) can be split: first, the system of order $j^* - 1$ is used to find θ^1, and then the vectors $(\nu_1^1, \ldots, \nu_{j^*}^1)$, $T_1^1, \ldots, T_{j^*}^1$ are calculated one after the other. A similar decomposition can be performed for the other systems in (4.24). Solving these systems one after the other, we find the coefficients θ^k, $T_1^k, \ldots, T_{j^*}^k$ and, therefore, polynomials (4.21), which yield the asymptotically s-optimal control (4.20).

The asymptotic approximations of the roots of system (4.18) thus constructed can be used to find the exact solution of this system and, therefore, the initial problem for the given value of the small parameter. This can be done by applying the refinement procedure described in [7]; i.e., the roots of system (4.18) are found by Newton's method with the initial approximation $z_s(\mu)$.

As in Section 3, we calculate the coefficients z_k $(k = \overline{1, s})$ and then $t_j^s(\delta) = \sum_{k=0}^s \delta^k t_j^k$ $(j = \overline{1, k^*})$ and $\theta_j^s(\delta) = \sum_{k=0}^s \delta^k \theta_j^k$ $(j = \overline{1, j^* - 1})$. The control $u^s(t)$ $(t \in T)$ of the form ((4.13) with $t_j = t_j^s(\delta)$ $(j = \overline{1, k^*})$ and $\theta_j = \theta_j^s(\delta)$ $(j = \overline{1, j^* - 1})$ is taken for the open loop solution to problem (2.1).

4.6. Realization of an Optimal Feedback

The realization of the globally optimal feedback for nonlinear systems is essentially the same as the method used to realize the locally optimal feedback for nonlinear systems described in Subsection 3.3. The difference is that the necessary procedures are more complicated. More precisely, at every current instant τ, the optimal solution to the piecewise linear (rather than linear) problem obtained for the preceding instant $\tau - h$ is corrected. Since the current states $x^*(\tau - h)$ and $x^*(\tau)$ of the nonlinear system are close to each other for small h, this operation (see [5]) can be done rather quickly. Then, the solution to the piecewise linear problem is corrected by the method described in Subsection 4.5, and the results are used to calculate the coefficients of the Taylor polynomials for the instants of switch of the asymptotically optimal control. These calculations do not require the solution of differential equations and do not take much time. Setting $\mu = \delta$ in the asymptotic expansion, we find the current value $u^*(\tau)$ of the optimal feedback realization.

To reduce the size of this paper, we omit the description of the procedures involved in this process. They can be reconstructed using the results obtained in [5] and Subsection 3.3 of this paper.

4.7. An Example

We illustrate the approach to the global optimization of nonlinear systems described in this paper by the problem of optimal damping of a simple pendulum:

$$\int_0^{10} u(t)dt \longrightarrow \min, \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = -\sin x_1 + u, \quad x_1(0) = 1.5, \quad x_2(0) = 0,$$
$$x_1(10) = x_2(10) = 0, \quad 0 \le u(t) \le 0.5, \quad t \in T = [0, 10], \tag{4.25}$$

in the domain $X = \{(x_1, x_2) : |x_1| < \pi/2\}$.

We will use two approximations of the nonlinear element $-\sin x_1$: (1) the linear approximation $-x_1$ ($x \in X$), and (2) the piecewise linear approximation $(1 - 4/\pi)x_1 + 1 - \pi/2$ for $x \in X_1 = \{(x_1, x_2) : \pi/4 < x_1 < \pi/2\}$ and $-x_1$, $x \in X_2 = \{(x_1, x_2\} : |x_1| < \pi/4\}$. The error of the linear approximation is $\delta_1 = 0.570796$, and for the piecewise linear approximation the error is $\delta_2 = 0.110721$.

Thus, in the first case, problem (4.25) is embedded in the family of problems

$$\int_0^{10} u(t)dt \longrightarrow \min, \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + \mu(x_1 - \sin x_1)/\delta_1 + u,$$
$$x_1(0) = 1.5, \quad x_2(0) = 0, \quad x_1(10) = x_2(10) = 0, \quad 0 \le u(t) \le 0.5, \quad t \in T. \tag{4.26}$$

In the second case, it is embedded in the family

$$\int_0^{10} u(t)dt \longrightarrow \min, \quad \dot{x}_1 = x_2,$$
$$\dot{x}_2 = \begin{cases} (1 - 4/\pi)x_1 + 1 - \pi/2 + \mu[(4/\pi - 1)x_1 - 1 + \pi/2 - \sin x_1]/\delta_2 + u, & x \in X_1, \\ -x_1 + \mu(x_1 - \sin x_1)/\delta_2 + u, & x \in X_2, \end{cases} \tag{4.27}$$
$$x_1(0) = 1.5, \quad x_2(0) = 0, \quad x_1(10) = x_2(10) = 0, \quad 0 \le u(t) \le 0.5, \quad t \in T.$$

In parameterized form, problem (4.27) is written as

$$\int_0^{10} u(t)dt \longrightarrow \min,$$
$$\dot{x}_1 = x_2, \quad \dot{x}_2 = (1 - 4/\pi)x_1 + 1 - \pi/2 + \mu[(4/\pi - 1)x_1 - 1 + \pi/2 - \sin x_1]/\delta_2 + u, \quad t \in [0, \theta_1[,$$
$$x_1(0) = 1.5, \quad x_2(0) = 0, \quad x_1(\theta_1) = \pi/4, \tag{4.28}$$
$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + \mu(x_1 - \sin x_1)/\delta_2 + u, \quad t \in [\theta_1, 10],$$
$$x_1(10) = x_2(10) = 0, \quad 0 \le u(t) \le 0.5, \quad t \in T.$$

Problem (4.28) corresponds to the basic problem

$$\int_0^{10} u(t)dt \longrightarrow \min,$$
$$\dot{x}_1 = x_2, \quad \dot{x}_2 = (1 - 4/\pi)x_1 + 1 - \pi/2 + u, \quad t \in [0, \theta_1[,$$
$$x_1(0) = 1.5, \quad x_2(0) = 0, \quad x_1(\theta_1) = \pi/4, \tag{4.29}$$
$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u, \quad t \in [\theta_1, 10],$$
$$x_1(10) = x_2(10) = 0, \quad 0 \le u(t) \le 0.5, \quad t \in T.$$

Table 3 presents the results of the open loop solution to problem (4.25). The trajectory of system (3.15) was constructed for the following two controls: (1) $u_1^0(t)$ $(t \in T)$, which is the optimal control for the basic linear problem

$$\int_0^{10} u(t)dt \longrightarrow \min, \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u, \quad x_1(0) = 1.5, \quad x_2(0) = 0,$$

$$x_1(10) = x_2(10) = 0, \quad 0 \le u(t) \le 0.5, \quad t \in T;$$

(2) $u_1^1(t)$ $(t \in T)$, which is the implementation of the asymptotically 1-optimal open loop control for problem (4.26) with the fixed $\mu = \delta_1$;

(3) $u_2^0(t)$ $(t \in T)$, which is the optimal control for the piecewise linear basic problem (4.29);

(4) $u_2^1(t)$ $(t \in T)$, which is the implementation of the asymptotically 1-optimal open loop control for problem (4.28) with the fixed $\mu = \delta_2$;

(5) $u^0(t, \delta)$ $(t \in T)$, which is the optimal open loop control for problem (4.25) constructed by the refinement procedure (see [7]) for the nonlinear problem (4.25).

In all cases, the control had the form

$$u(t) = \begin{cases} 0, & t \in [0, t_1[\cup [t_2, t_3[\cup [t_4, 10[, \\ 0.5, & t \in [t_1, t_2[\cup [t_3, t_4[. \end{cases} \tag{4.30}$$

Table 3 presents the switching points of these controls, the instant θ_1 of transition between the linearity domains for the piecewise linear approximation, the value of the objective functional, and the terminal state of system (4.25).

Table 3

Control $(t \in T)$	Switching points	Instant θ_1	Value of the objective functional	Terminal state
$u_1^0(t)$	0.722734	–	1.696124	0.017095
	2.418858			–0.516580
	7.005920			
	8.702044			
$u_1^1(t)$	1.007916	–	1.489935	–0.012199
	2.517443			–0.055845
	7.547231			
	9.017574			
$u_2^0(t)$	1.078442	1.233553	1.509974	–0.013850
	2.593439			–0.111839
	7.341674			
	8.876624			
$u_2^1(t)$	1.064668	1.218658	1.496074	–0.001177
	2.573226			–0.008707
	7.553250			
	9.036841			
$u^0(t, \delta)$	1.064547	–	1.496228	10^{-8}
	2.573684			10^{-8}
	7.566068			
	9.049386			

Table 4

Control ($t \in T$)	Switching points	Value of the objective functional	Terminal state
$u^{1*}(t)$	1.3 2.7 7.789350 9.287743	1.449193	0.000016 –0.001192
$u_{10}^{1*}(t)$	1.3 2.7 7.717981 9.257927	1.469973	–0.003723 –0.002823
$u_{50}^{1*}(t)$	1.3 2.7 7.778486 9.278203	1.449858	–0.000074 –0.001229
$u_{100}^{1*}(t)$	1.3 2.7 7.778420 9.278330	1.449955	10^{-6} –0.001202
$u_{300}^{1*}(t)$	1.3 2.7 7.778403 9.278366	1.449982	0.000014 –0.001133

Now, consider the construction of a closed loop control for problem (4.25). As in Section 3, we feed the control $u^{1*}(t)$ produced by the 1-optimal controller ($h = 0.01$) to system (4.25). The required values of the functions $x_1(t)$ and $\psi_1(t)$ ($t \in T^\tau$, $\tau \in T_h$) were found by the method described in [13]. The control $u^{1*}(t)$ ($t \in T$) had the form (4.30) with the switching points 1.06, 2.58, 7.567630, 9.039940, and the instant of transition between the linearity domains was $\theta_1^* = 1.21$. At $t^* = 10$, the trajectory of system (4.25) governed by the control $u^{1*}(t)$ ($t \in T$) reached the state (–0.000129, –0.000455), and the value of the objective functional was 1.496155.

Consider the behavior of the system subjected to the perturbation $w^*(t) = 0.4\sin(3t)$ for $t \in [0, 7[$ and $w^*(t) \equiv 0$ for $t \geq 7$, which is not perceived by the controller:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\sin x_1 + u + w^*(t), \quad x_1(0) = 1.5, \quad x_2(0) = 0. \tag{4.31}$$

Trajectories of system (4.31) were constructed for various realizations of the optimal feedback: (1) $u^{1*}(t)$ ($t \in T$) produced by the 1-optimal controller when the required values of $x_1(t)$ and $\psi_1(t)$ ($t \in T^\tau$, $\tau \in T_h$) were calculated by the method described in [13]; (2) $u_N^{1*}(t)$ ($t \in T$) produced by the 1-optimal controller when the required values of $x_1(t)$ and $\psi_1(t)$ ($t \in T^\tau$, $\tau \in T_h$) were calculated by the mean rectangular quadrature formula with N nodes ($N = 10$, 50, 100, and 300). The controls had the form (4.30). For each of them, the instant of transition was $\theta_1^* = 1.39$.

Table 4 presents the instants of switching of the above controls, the corresponding values of the objective functional, and the terminal states of system (4.31).

Remark. In this paper, the set X in the stationary nonlinear system was independent of time. The efficiency of our approach can be improved by considering sets $X(t)$ ($t \in T$) that describe neighborhoods of a certain reference trajectory containing the optimal trajectories ($x^0(t) \in X(t)$, $t \in T$). Then, the approximation error $\delta(t)$ ($t \in T$) can be reduced for the "operating" domains of the phase space, and "nonoperating" domains can be excluded from the consideration.

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