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CONTROL THEORY

Synthesis of Optimal Systems of Indirect Control

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1. We consider the system

$$\dot{x} = A(t)x + b(t)u, \quad x(t_*) = x_0, \tag{1}$$

which is controlled by the first-order actuator

$$\dot{u} = au + v, \quad u(t_*) = u_0 \tag{2}$$

on an interval $T = [t_*, t^*]$. Here, $x = x(t)$ is the n-dimensional vector state of the system, $u = u(t)$ is the control action, $v = v(t)$ is the control signal, $A(t) \in \mathbf{R}^{n \times n}$ and $b(t) \in \mathbf{R}^n$ $(t \in T)$ are piecewise continuous functions, a is a scalar, and x_0 and u_0 are initial states.

The output and input signals of actuator (2) are subject to the constraints

$$|u(t)| \le L, \quad |v(t)| \le M, \quad t \in T \quad (|a|L \le M).$$

For system (1) controlled by actuator (2) via piecewise continuous control signals, we consider the terminal optimal control problem

$$\begin{gathered} c'x(t^*) \to \max, \quad \dot{x} = A(t)x + b(t)u, \\ x(t_*) = x_0, \quad \dot{u} = au + v, \quad u(t_*) = u_0, \\ |u(t)| \le L, \quad |v(t)| \le M, \quad t \in T, \quad Hx(t^*) = g \\ (g \in \mathbf{R}^m, H \in \mathbf{R}^{m \times n}, \operatorname{rank} H = m < n). \end{gathered} \tag{3}$$

We introduce the concept of a closed-loop solution to problem (3). Assume that the state of system (1) in the control process is measured at times $\tau \in T_{h_0} = \{t_*, t_* + h_0, \dots, t^* - h_0\}$ $\left(\text{where } h_0 = \dfrac{t^* - t_*}{N_0} \text{ and } N_0 \text{ is a positive integer}\right)$. Problem (3) is embedded in the family of problems

$$\begin{gathered} c'x(t^*) \to \max, \quad \dot{x} = A(t)x + b(t)u, \\ x(\tau) = z, \quad \dot{u} = au + v, \quad u(\tau) = y, \\ Hx(t^*) = g, \quad |u(t)| \le L, \quad |v(t)| \le M, \\ t \in T(\tau) = [\tau, t^*], \end{gathered} \tag{4}$$

which depend on the position $(\tau; z, y)$. Let $v^0(t|\tau, z, y)$, $t \in T(\tau)$, be the optimal open-loop control signal in problem (4) for the position $(\tau; z, y)$, and let G_τ be the set of all pairs (z, y) for which problem (4) has a solution.

The function

$$\begin{gathered} v^0(\tau, z, y) = (v^0(t|\tau, z, y), t \in [\tau, \tau + h_0[), \\ (z, y) \in G_\tau, \quad \tau \in T_{h_0} \end{gathered}$$

is called an optimal feedback control signal, and its construction is called the synthesis of an optimal feedback (the synthesis of an optimal system).

One method for implementing an optimal feedback can be described as follows. The method is based on an analysis of using the optimal feedback in the control process. Assume that an optimal feedback $v^0(t, x, u)$, $(x, y) \in G_t$, $t \in T_{h_0}$ has been constructed. Consider the corresponding closed-loop physical prototype of system (1):

$$\begin{gathered} \dot{x} = A(t)x + b(t)u + w, \quad x(t_*) = x_0, \\ \dot{u} = au + v^0(t, x, u), \quad u(t_*) = u_0, \end{gathered} \tag{5}$$

where $w = w(t, x, u)$, $(x, u) \in G_t$, $t \in T$ is a disturbance, i.e., the set of terms corresponding to the unknown disturbances and inaccuracies in simulation. Assume that, in every process $x = x(t)$, $u = u(t)$, $t \in T$, the disturbance is a piecewise continuous n-dimensional vector function $w(t) = w(t, x(t), u(t))$, $t \in T$.

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The trajectory of system (5) is the solution to the linear system

$$\dot{x} = A(t)x + b(t)u + w, \quad x(t_*) = x_0,$$
$$\dot{u} = au + v^0(t), \quad u(t_*) = u_0$$

under the control signal $v^0(t) = v^0(t|\tau, x(\tau), u(\tau))$, $t \in [\tau, \tau + h_0[$, $\tau \in T_{h_0}$.

Consider a control process. Suppose that an unknown piecewise continuous disturbance $w^*(t)$, $t \in T$ is realized in this control process. That disturbance generates a trajectory $x^*(t)$, $t \in T$ and a control action $u^*(t)$, $t \in T$ in system (5) that satisfy the identities

$$\dot{x}^*(t) \equiv A(t)x^*(t) + b(t)u^*(t) + w^*(t), \quad x^*(t_*) = x_0,$$
$$\dot{u}^*(t) \equiv au^*(t) + v^0(t, x^*(t), u^*(t)), \quad u^*(t_*) = u_0.$$

It can be seen that the optimal feedback is not used completely [for all $(x, u) \in G_t$, $t \in T_{h_0}$] in the control process, but only its values $v^*(t) = v^0(t|\tau, x^*(\tau), u^*(\tau))$, $t \in [\tau, \tau + h_0[$, $\tau \in T_{h_0}$ along continuous curves $x^*(t)$, $u^*(t)$, $t \in T$ are used in the control. The function $v^*(t)$, $t \in T$ is called an optimal feedback realization in the control process. If the values of $v^*(t)$, $t \in [\tau, \tau + h_0[$ at every current instant of time $\tau \in T_{h_0}$ are calculated in a time not exceeding h_0, then we say that the optimal feedback realization is constructed in real time. A device capable of doing this operation is called an optimal controller. Thus, the synthesis of an optimal feedback control signal is reduced to the design of an operation algorithm for an optimal controller.

The operation algorithm for an optimal controller consists of two procedures. Prior to the beginning of the control process, the initial procedure constructs an open-loop solution to problem (3), which is used to control the actual system on $[t_*, t_* + h_0[$. The general procedure receives realizations of the optimal feedback $v^*(t)$, $t \in [\tau, \tau + h_0[$, $\tau \in T_{h_0} \backslash t_*$.

2. Along with (3), we consider the maximum excitation problem

$$\begin{gathered} c'(\nu)x(t^*) \to \max, \quad \dot{x} = A(t)x + b(t)u, \\ x(t_*) = x_0, \quad \dot{u} = au + v, \quad u(t_*) = u_0, \\ |u(s)| \le L, \quad s \in S_h = \{t_* + h, \ldots, t^*\}, \\ |v(t)| \le M, \quad t \in T, \end{gathered} \tag{6}$$

in which $c(\nu) = c - H'\nu$, $\nu \in \mathbf{R}^m$ is a Lagrange vector, $h = \dfrac{t^* - t_*}{N}$, and $N > 0$.

Let $v^0(t)$ and $v^0(t|\nu)$ $(t \in T)$ be the optimal open-loop control signals in problems (3) and (6), respectively. According the Lagrange multiplier method, there exists a vector $\nu^0 \in \mathbf{R}^m$ such that $v^0(t) \equiv v^0(t|\nu^0)$, $t \in T$.

The optimal vector ν^0 is constructed in two stages. First, the formula

$$\nu^{(p+1)} = \nu^{(p)} + \theta_p(Hx(t^*|\nu^{(p)}) - g) \tag{7}$$

is used to iteratively improve the optimal vector of potentials $\nu^{(1)}$ [2] in terminal problem (3) directly controlled by the bounded control $|u(t)| \le L$, $t \in T$. Here, θ_p is the step chosen according to any gradient method and $x(t^*|\nu)$ is the optimal terminal state in problem (6) at $\nu = \nu^{(p)}$. Then, the required accuracy in the optimality conditions and the terminal constraint is achieved by applying a refinement procedure, which solves the refinement equations by Newton's method.

3. To implement the first stage, we construct an efficient algorithm for computing an open-loop solution to problem (6), which is an optimal control problem with a state constraint.

Problem (6) is solved in the class of discrete-time control signals with a quantization period h: $v(t) \equiv v(\tau)$, $t \in [\tau, \tau + h[$, $\tau \in T_h$. In this case, it is equivalent to the linear programming problem (LP)

$$\begin{gathered} \sum_{t \in T_h} c(t)v(t) \to \max; \\ L_*(s) \le \sum_{t = t_*}^{s-h} d(s-t)v(t) \le L^*(s), \\ s \in S_h; \quad |v(t)| \le M, \quad t \in T_h. \end{gathered} \tag{8}$$

Here, $c(t) = \int_t^{t+h} \psi_{n+1}(\vartheta)d\vartheta$ for $t \in T_h$ and $\psi_{n+1}(t)$ $(t \in T)$ is found from the dual system

$$\dot{\psi}' = -\psi'A(t), \quad \dot{\psi}_{n+1} = -a\psi_{n+1} - \psi'b(t),$$
$$\psi(t^*) = c(\nu), \quad \psi_{n+1}(t^*) = 0,$$

$$d(t) = \int_0^h \Phi(t-\vartheta)d\vartheta, \quad t \in T_h;$$
$$L_*(s) = -L - \Phi(s)u_0, \quad L^*(s) = L - \Phi(s)u_0,$$
$$s \in S_h;$$

$\Phi(t)$, $t \ge 0$, is the fundamental matrix of solutions to (2): $\dot{\Phi} = a\Phi$, $\Phi(0) = 1$.

For small h, problem (8) is a large-scale LP problem with a special matrix of basic constraints. The general LP methods fail to solve this problem, since they ignore its specific features. Problem (8) is solved by the adaptive LP method [1], with the specific features of the problem taken into account as far as possible.

The basic tool used in the method [1] is the support. The empty support is defined as the pair $K_{\rm sup} = \{S_{\rm sup} = \emptyset, T_{\rm sup} = \emptyset\}$. A nonempty support $K_{\rm sup} = \{S_{\rm sup} \subset S_h, T_{\rm sup} \subset T_h\}$ has the form $S_{\rm sup} = \bigcup_{l=1}^{l^*} S_l$, $S_l = \{s_l, s_l + h, \ldots, s_l\}$; $T_{\rm sup} = \bigcup_{l=1}^{l^*} (T_l \cup \tau_l)$, $T_l = S_l \backslash s^l$, $s^{l-1} \le \tau_l \le s_l$, $l = 1, 2, \ldots, l^*$ $(s_0 = t_*, s_{l^*+1} = t^*)$.

The support $K_{\rm sup}$ is associated with the following objects:

(i) the function of potentials $\nu_h(s)$, $s \in S_h$:

$$\nu_h(s) = 0, \quad s \in S_{\rm n} = S_h \backslash S_{\rm sup};$$

$$\nu_h(s) = \frac{c(t_*(s)) - c(t^*(s))\Phi(t^*(s) - t_*(s))}{d(s - t_*(s))}, \quad s \in S_{\rm sup};$$

where $t_*(s) < s$ is the moment in $T_{\rm sup}$ nearest to s on the left and $t^*(s) \ge s$ is the moment nearest to s on the right;

(ii) the cocontrol

$$\Delta_h(t) = 0, \quad t \in T_{\rm sup}; \quad \Delta_h(t) = c(t), \quad t \ge s^{l^*};$$

$$\Delta_h(t) = c(t) - c(\tau_l)\Phi(\tau_l - t), \quad s^{l-1} \le t < s_l, \quad l = 1, 2, \ldots, l^*;$$

(iii) the pseudosignal $\omega(t)$ $(t \in T)$ and the pseudoaction $\zeta(s)$, $s \in T$. The nonsupport values of the pseudosignal and the support values of the pseudoaction are

$$\omega(t) = M\,\mathrm{sgn}\Delta_h(t) \text{ if } \Delta_h(t) \ne 0;$$

$$\omega(t) \in [-M, M] \text{ if } \Delta_h(t) = 0, \quad t \in T_{\rm n} = T_h \backslash T_{\rm sup};$$

$$\zeta(s) = L\,\mathrm{sgn}\nu_h(s) \quad \text{if} \quad \nu_h(s) \ne 0;$$

$$\xi(s) \in [-L, L] \quad \text{if} \quad \nu_h(s) = 0 \quad \text{if} \quad s \in S_{\rm sup}.$$

The support values of the pseudosignal and the pseudoaction are calculated by the formulas

$$\omega(\tau_l) = \frac{\zeta(s_l) - \Phi(s_l - s^{l-1})\zeta(s^{l-1}) - \sum\limits_{s^{l-1} \le t < s_l,\, t \ne \tau_l} d(s_l - t)\omega(t)}{d(s_l - \tau_l)}, \quad \zeta(s^0) = u_0,$$

$$\omega(t) = \frac{\zeta(t+h) - \Phi(h)\zeta(t)}{d(h)}, \quad t \in T_l, \quad l = 1, 2, \ldots, l^*;$$

$$\zeta(s) = \Phi(s - s^{l-1})\zeta(s^{l-1}) + \sum_{s^{l-1} \le t < s} d(s-t)\omega(t), \quad s^{l-1} < s < s_l, \quad l = 1, 2, \ldots, l^* + 1.$$

The following statement follows from the results of [1].

ε-Maximum principle. For any $\varepsilon \ge 0$, an admissible control signal $v(t)$ $(t \in T)$ and an admissible control action $u(t)$ $(t \in T)$ are optimal if and only if there exists a support $K_{\rm sup}$ such that the following relations are fulfilled for some of its associated elements $\nu_h(s)$, $s \in S_h$; $\Delta_h(t)$, $t \in T_h$:

(i) the ε-maximum condition with respect to the control action

$$\nu_h(s)u(s) = \max_{|u| \le L} \nu_h(s)u - \varepsilon_u(s), \quad s \in S_{\rm sup};$$

(ii) the ε-maximum condition with respect to the control signal

$$\Delta_h(t)v(t) = \max_{|v| \le M} \Delta_h(t)v - \varepsilon_v(t), \quad t \in T_{\rm n};$$

(iii) the ε-accuracy condition

$$\sum_{s \in S_{\rm sup}} \varepsilon_u(s) + \sum_{t \in T_{\rm n}} \varepsilon_v(t) \le \varepsilon.$$

A support $K_{\rm sup}$ satisfying the ε-maximum principle for $\varepsilon = 0$ is called optimal. The ε-maximum principle implies that a support $K_{\rm sup}$ is optimal if and only if there exist associated elements such that $|\zeta(s)| \le L$ for $s \in S_{\rm n}$ and $|\omega(t)| \le M$ for $t \in T_{\rm sup}$. Moreover, $v^0(t) = \omega(t)$ for $t \in T$ and $u^0(s) = \zeta(s)$ for $s \in S$.

The method used for solving problem (6) is an iterative process of changing supports, which begins with $K_{\rm sup} = \emptyset$ and ends with an optimal support $K^0_{\rm sup}$. First the "outliers" $|\zeta(s_0)| > L$ of the pseudoaction and then the violations of the direct constraints $|\omega(\tau_l)| > M$ are eliminated at the iterations of the method.

The following information is stored before the beginning of a current iteration step: (1) the number l^* of support segments; (2) the endpoints s_l and s^l of the segments S_l; (3) the instants of time τ_l, $l = 1, 2, \ldots, l^*$;

(4) the values $\nu(s_l)$ and $\nu(s^l)$, $l = 1, 2, \dots, l^*$; (5) $\omega(\tau_l)$, $l = 1, 2, \dots, l^*$; (6) $\zeta_l = \zeta(s_l)$, $l = 1, 2, \dots, l^*$, $\zeta(t^*)$; (7) the sets of nonsupport zeros of the cocontrol $T^l_{n0} = \{t \in [s^{l-1}, s_l[\cap T_h$: $\Delta_h(t-h)\Delta_h(t) < 0\}$, $l = 1, 2, \dots, l^* + 1$; (8) the numbers $\gamma^l = \operatorname{sgn}\Delta(s^{l-1})$ if $s^{l-1} \neq \tau_l$ and $\gamma^l = \operatorname{sgn}\Delta(s^{l-1} + h)$ if $s^{l-1} = \tau_l$, $l = 1, 2, \dots \dots, l^* + 1$; (9) $\Phi(s_l - t)$, $\psi_{n+1}(t)$, $t \in T^l_0 = T^l_{n0} \cup \{s^{l-1}, \tau_l, s_l\}$, $l = 1, 2, \dots, l^* + 1$; and (10) the numbers $p_l = \sum\limits_{s^{l-1} \le t < s_l,\, t \neq \tau_l} d(s_l - t)\omega(t)$, $l - 1, 2, \dots, l^*$.

The iteration step at which the outlier $\xi(s_0)$ is eliminated consists of the following operations:

(1) Calculate $\Delta\nu(s)$ ($s \in S_h$) and $\Delta\delta(t)$ ($t \in T_h$) for subcases (I) $\tau_{l_0} < s_0 < s_{l_0}$ and (II) $s^{l_0-1} < s_0 \le \tau_{l_0}$:

$$\text{(I)}\ \Delta\nu(s_0) = \operatorname{sgn}\zeta(s_0),\quad \Delta\nu(s_{l_0}) = -\Delta\nu(s_0)\Phi(s_0 - s_{l_0}),$$

$$\Delta\nu(s) = 0,\quad s \in S_h\backslash\{s_0, s_{l_0}\};$$

$$\Delta\delta(t) = \Delta\nu(s_0)d(s_0 - t),\quad s_0 \le t < s_{l_0},$$
$$\Delta\delta(t) = 0,\quad t < s_0,\quad t \ge s_{l_0};$$

$$\text{(II)}\qquad \Delta\nu(s_0) = \operatorname{sgn}\zeta(s_0),$$
$$\Delta\nu(s^{l_0-1}) = -\Delta\nu(s_0)\Phi(s_0 - s^{l_0-1}),$$

$$\Delta\nu(s) = 0,\quad s \in S_h\backslash\{s_0, s^{l_0-1}\};$$

$$\Delta\delta(t) = \Delta\nu(s_0)d(s_0 - t),\quad s^{l_0-1} \le t < s_0,$$
$$\Delta\delta(t) = 0,\quad t < s^{l_0-1},\quad t \ge s_0.$$

(2) Calculate the steps $\sigma_\nu(s) = -\dfrac{\nu_h(s)}{\Delta\nu(s)}$ for $s \in S_{\text{sup}}$ such that $\nu_h(s)\Delta\nu(s) < 0$, and $\sigma_\delta(t) = -\dfrac{\Delta_h(t)}{\Delta\delta(t)}$ for $t \in T_{\text{n}}$ such that $\Delta_h(t)\Delta\delta(t) < 0$. The steps are arranged in increasing order: $0 < \sigma^1 < \dots < \sigma^{k^0}$. A long dual step $\sigma^* = \sigma^{k^*}$ is found [1]: $\alpha^{k^*} < 0$, $\alpha^{k^*+1} \ge 0$, where α^k is the rate of change in the performance criterion of the problem dual to (6) on the interval $[\sigma^{k-1}, \sigma^k[$ ($\sigma^0 = 0$): $\alpha^1 = -\rho(\zeta(s_0), [-L, L])$,[1] $\alpha^k = \alpha^{k-1} + \Delta\alpha^k$, and $\Delta\alpha^k = 2M|\Delta\delta(t_k)|$ if $\sigma^k = \sigma_\delta(t_k)$; $\Delta\alpha^k = 2L|\Delta\nu(s_k)|$ if $\sigma^k = \sigma_\nu(s_k)$.

The main feature of problem (6) is manifested here: the values of $\Delta\nu(s)$ ($s \in S_h$) on the set $s \in S_{\text{sup}}$ are zero everywhere except for no more than two points. Thus, no more than two numbers are calculated while we determine $\sigma_\nu(s)$, $s \in S_{\text{sup}}$. Thus, the labor-consuming handling (for small h) of the state constraints is eliminated.

A fast procedure for calculating $\sigma_\delta(t)$ was described in [2]. It is based on the fact that $\sigma_1, \sigma_2, \dots, \sigma^{k^*}$ belong to the neighborhood of the nonsupport zeros of the cocontrol, in which case we do not need to calculate all $\sigma_\delta(t)$, $t \in T_{\text{n}}$. The value of to σ^* is rapidly calculated by using (7) and (9). The feature of calculating $\sigma_\nu(s)$ ($s \in S_{\text{sup}}$) makes this method for solving problem (6) equivalent (in computational work) to the method of [2] for a problem without state constraints.

(3) A new support $\bar{K}_{\text{sup}} = \{\bar{S}_{\text{sup}}, \bar{T}_{\text{sup}}\}$ is constructed, depending on the following situations: (a) $\sigma^* = \sigma_\delta(\tau_*)$, (b) $\sigma^* = \sigma_\nu(s_*)$; (a) $\bar{S}_{\text{sup}} = S_{\text{sup}} \cup s_0$, $\bar{T}_{\text{sup}} = T_{\text{sup}} \cup \tau_*$; (b) $\bar{S}_{\text{sup}} = (S_{\text{sup}}\backslash s_*) \cup s_0$, $\bar{T}_{\text{sup}} = T_{\text{sup}}$. The support sets $\bar{S}_{\text{sup}}$ and $\bar{T}_{\text{sup}}$ are constructed taking into account their structure. Information (1)–(10) is updated for the following iteration step.

While the outlier $\omega(\tau_l)$ is eliminated, we construct

$$\Delta\delta(\tau_l) = \operatorname{sgn}\omega(\tau_l);\quad \Delta\delta(t) = \Delta\delta(\tau_l)\Phi(\tau_l - t),$$
$$s^{l-1} - h \le t < s_l,$$

$$\Delta\delta(t) = 0,\quad t < s^{l-1} - h,\quad t \ge s_l;$$

$$\Delta\nu(s) = 0,\quad s \in S_h\backslash\{s_l, s^{l-1}\};\quad \Delta\nu(s_l) = -\frac{\Delta\delta(\tau_l)}{d(s_l - \tau_l)},$$

$$\Delta\nu(s^{l-1}) = \frac{\Delta\delta(\tau_l)}{d(s^{l-1} - \tau_l)}.$$

The value $\alpha^1 = -\rho(\omega(\tau_l), [-M, M])$ is taken into account at stage (2). A new support is constructed at stage (3) according to the following rules: (a) $\bar{S}_{\text{sup}} = S_{\text{sup}}$, $\bar{T}_{\text{sup}} = (T_{\text{sup}} \cup \tau_*)\backslash\tau_l$; and (b) $\bar{S}_{\text{sup}} = S_{\text{sup}}\backslash s_*$, $\bar{T}_{\text{sup}} = T_{\text{sup}}\backslash t_l$.

4. The refinement procedure can be described as follows. The defining elements of the control signal $\nu^0(t)$ ($t \in T$) in problem (3) are the Lagrange vector ν and the switching points

$$t_1^1, \dots, t^1_{n^1}, s_1, s^1, t_1^2, \dots, t^2_{n^2}, s_2, \dots, s^{l-1},$$
$$t_1^l, \dots, t^l_{n^l}, s_l, s^l, \dots, s^{l^*}, t_1^{l^*+1}, \dots, t^{l^*+1}_{n^{l^*}+1}.$$

Then the control signal takes the form

$$v^0(t) = \begin{cases} (-1)^i\gamma^l M, & t \in [t_i^l, t_{i+1}^l[\\ i = 0, 1, \dots, n^l, & l = 1, 2, \dots, l^* + 1 \\ \gamma^{l+1}aL, & t \in [s_l, s^l[, \quad l = 1, 2, \dots, l^*, \end{cases} \qquad (9)$$

[1] $\rho[c, [a, b])$ is the distance from the point c to $[a, b]$.

where $\gamma^l = \operatorname{sgn}\lambda(s^{l-1}+0)$, $l = 1, 2, \ldots, l^*+1$; $\lambda(t) = 0$, $t \in [s_l, s^l]$, $l = 1, 2, \ldots, l^*$; and $\lambda(t)$, $t \in]s^{l-1}, s_l[$, $l = 1, 2, \ldots, l^*+1$ is the solution to the dual system[2]

$$\dot{\lambda} = -a\lambda - \psi' b(t), \quad \dot{\psi}' = -\psi' A(t),$$
$$\psi(t^*) = c(\nu), \quad \lambda(t^*) = 0,$$
$$\lambda(s_l) = 0, \quad l = 1, 2, \ldots, l^*.$$

The defining elements satisfy the system of equations

$$R_0(y_1, y_2, \ldots, y_{l^*+1}) = 0,$$
$$R_l(\nu, y_l) = 0, \quad l = 1, 2, \ldots, l^*+1, \tag{10}$$

where[3] $y_1 = (t_1^1, t_2^1, \ldots, t_{n^1}^1, s_1)$; $y_l = (s^{l-1}, t_1^l, \ldots, t_{n^l}^l, s_l)$ for $l = 2, 3, \ldots, l^*$; $y_{l^*+1} = (s^{l^*}, t_1^{l^*+1}, \ldots, t_{n^{l^*+1}}^{l^*+1})$;

$$R_0(y_1, y_2, \ldots, y_{l^*+1})$$
$$= \sum_{l=1}^{l^*+1}\sum_{i=0}^{n^l}(-1)^i\gamma^l M\int_{t_i^l}^{t_{i+1}^l} g(\vartheta)d\vartheta$$
$$+\sum_{l=1}^{l^*} a\gamma^{l+1}L\int_{s_l}^{s^l} g(\vartheta)d\vartheta + G(t_*)x_0 + g(t_*)u_0 - g,$$

$$R_l(y_l)$$
$$= \begin{pmatrix} \displaystyle\int_{s^{l-1}}^{s_l}\psi_c'(\vartheta)b(\vartheta)\Phi(\vartheta - s^{l-1})d\vartheta \\ \displaystyle\int_{t_i^l}^{s_l}\psi_c'(\vartheta)b(\vartheta)\Phi(\vartheta - t_i^l)d\vartheta, \quad i = 1, 2, \ldots, n^l \\ \gamma^{l+1}L - \Phi(s_l - s^{l-1})\gamma^l L \\ \displaystyle +\sum_{i=0}^{n^l}(-1)^i\gamma^l M\int_{t_i^l}^{t_{i+1}^l}\Phi(s_l - \vartheta)d\vartheta \end{pmatrix} = 0,$$
$$l = 1, 2, \ldots, l^*+1;$$

$$\dot{G} = -GA(t), \quad G(t^*) = H,$$
$$\dot{g} = -ag - Gb(t), \quad g(t^*) = 0.$$

[2] The function $\lambda(t)$, $t \in T$, is continuous on the defining elements $v_0(t)$, $t \in T$.

[3] The equation $R_1(y_1) = 0$ does not contain the first equation, and the equation $R_{l^*+1}(y_{l^*+1}) = 0$ does not contain the last equation.

System (10) is solved by Newton's method, which generates approximations $(\nu^{(k)}, y^{(k)})$, starting with $\nu^{(0)}$ determined by (7) and with the elements of T_0^l ($l = 1, 2, \ldots, l^*+1$) found by solving problem (6) for $\nu = \nu^{(0)}$.

5. Assume that the operation algorithm for an optimal controller has been constructed. Assume that the controller has operated over the interval $[t_*, \tau[$ and produced control signals $v^*(t)$, $t \in [t_*, \tau[$ generating a control action $u^*(t)$, $t \in [t_*, \tau[$, which, together with a disturbance $w^*(t)$, $t \in [t_*, \tau]$, transferred the control system at the current time τ to the state $x^*(\tau)$. To determine $v^*(t)$ for $t \in [\tau, \tau + h_0[$, the optimal controller must rapidly find the open-loop solution to problem (4) at $(\tau; x^*(\tau), u^*(\tau))$. On the interval $[\tau - h_0, \tau[$, the optimal controller has solved a similar problem, in which the process started at the time $\tau - h_0$ at the state $x^*(\tau - h_0)$ with control action $u^*(\tau - h_0)$. Its solution (defining elements) $(\nu(\tau - h_0), y_l(\tau - h_0))$, $l = 1, 2, \ldots, l^*(\tau - h_0) + 1$ have been constructed.

Let $y_1^1(\tau - h_0)$ be the first element of the vector $y_1(\tau - h_0)$. If $\tau < y_1^1(\tau - h_0)$, then the defining elements obtained at the preceding instant of time are used as the initial approximation for Newton's method:

$$\nu^{(1)}(\tau) = \nu(\tau - h_0), \quad y_l^{(1)}(\tau) = y_l(\tau - h_0),$$
$$l = 1, 2, \ldots, l^*(\tau - h_0) + 1.$$

Moreover,

$$R_l(\nu^{(1)}(\tau), y^{(1)}(\tau)) = 0,$$
$$l = 1, 2, \ldots, l^*(\tau - h_0) + 1;$$

$$R_0(y^{(1)}(\tau)) = G(\tau)x^*(\tau) + g(\tau)u^*(\tau)$$
$$-G(\tau - h_0)x^*(\tau - h_0) - g(\tau - h_0)u^*(\tau - h_0)$$
$$-v^*(\tau - h_0)\int_{\tau - h_0}^{\tau} g(\vartheta)d\vartheta.$$

When $\tau \geq y_1^1(\tau - h_0)$, we set $\nu^{(1)}(\tau) = \nu(\tau - h_0)$, $y_1^{(1)}(\tau) = (y_1^2(\tau - h_0), \ldots, y_1^{n^1+1}(\tau - h_0))$, $y_l^{(1)}(\tau) = y_l(\tau - h_0)$, $l = 2, 3, \ldots, l^*(\tau - h_0) + 1$; $\gamma^1(\tau) = -\gamma^1(\tau - h_0)$ if $y_1^1(\tau - h_0) = t_1^1(\tau - h_0)$; $\gamma^1(\tau)$ is not defined if $y_1^1(\tau - h_0) = s_1(\tau - h_0)$ (the control action is on the boundary of the constraint); and $\gamma^1(\tau) = \gamma^2(\tau - h_0)$ if $y_1^1(\tau - h_0) = s^1(\tau - h_0)$. The values of $R_0(y^{(1)}(\tau))$ and $R_l(\nu^{(1)}(\tau), y^{(1)})$, $l = 1, 2,$

…, $l^*(\tau)$, + 1 are constructed as in the previous case, and the dimension of R_1 reduces by unity.

Because of the proximity of the problems for τ and $\tau - h_0$ when h_0 is small, the initial approximations $(\nu^{(1)}(\tau), y^{(1)}(\tau))$ are close to the exact solution $(\nu(\tau), y(\tau))$ of Eq. (10). A small number of Newton iterations are required for constructing a highly accurate solution.

By using the defining elements $(\nu(\tau), y(\tau))$ and formula (9), we construct the optimal open-loop control signal $v^0(t|\tau, x^*(\tau), u^*(\tau))$, $t \in [\tau, \tau + h_0[$, which is used on $[\tau, \tau + h_0[$ as a realization of the optimal feedback.

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